

THE AUSLANDER-REITEN TRANSLATE ON MONOMIAL RINGS

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ABSTRACT. For \mathbf{t} in \mathbb{N}^n , E. Miller has defined a category of positively \mathbf{t} -determined modules over the polynomial ring S in n variables. We consider the Auslander-Reiten translate, $\mathcal{N}_{\mathbf{t}}$, on the (derived) category of such modules. A monomial ideal I is \mathbf{t} -determined if each generator $\mathbf{x}^{\mathbf{a}}$ has $\mathbf{a} \leq \mathbf{t}$. We compute the multigraded cohomology and betti spaces of $\mathcal{N}_{\mathbf{t}}^k(S/I)$ for every iterate k and also the S -module structure of these cohomology modules. This comprehensively generalizes results of Hochster and Gräbe on local cohomology of Stanley-Reisner rings.

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1. INTRODUCTION

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring over a field K . On the category of graded projective S -modules there is the standard duality $D = \text{Hom}_S(-, S)$. In the case of \mathbb{N}^n -graded S -modules, on the subcategory of square free modules there is also a more combinatorially defined Alexander duality introduced by E. Miller, [8], and T. Römer, [10]. More generally Miller for any \mathbf{t} in \mathbb{N}^n defines a subcategory of \mathbb{N}^n -graded modules, the category of positively \mathbf{t} -determined modules. They have the property that they are determined (in a specified way) by their S -module structure for degrees in the interval $[\mathbf{0}, \mathbf{t}]$. On this category Miller also defines an Alexander duality $A_{\mathbf{t}}$. Considering standard duality in this setting it will be convenient to consider a twisted version. Replace S with an injective resolution I , and consider the functor $D_{\mathbf{t}} = \text{Hom}_S(-, I(-\mathbf{t}))$, which becomes a duality (up to quasi-isomorphism) on the category of chain complexes of positively \mathbf{t} -determined S -modules.

The main object of this paper is to study the composition $A_{\mathbf{t}} \circ D_{\mathbf{t}}$ of the two functors and its iterates. This composition appears in various guises in the literature. Firstly, if M is any positively \mathbf{t} -determined module, the cohomology of $A_{\mathbf{t}} \circ D_{\mathbf{t}}(M)$ is simply the local cohomology modules of M , slightly rearranged. Secondly, the category of positively \mathbf{t} -determined modules is equivalent to the category of modules over the incidence algebra of the poset $[\mathbf{0}, \mathbf{t}]$, which is an artin algebra. With this identification $A_{\mathbf{t}} \circ D_{\mathbf{t}}$ is the Auslander-Reiten translate on the derived category of modules over this incidence algebra, see [6, p.37]. In this derived setting $A_{\mathbf{t}} \circ D_{\mathbf{t}}$ may also be identified as the Nakayama functor, [1, p.126], and this is the terminology we shall use for this composition. Thirdly, in the case when \mathbf{t} is equal to $\mathbf{1} = (1, 1, \dots, 1)$, the case of square free modules, the second author, in [4], considers $A_{\mathbf{t}} \circ D_{\mathbf{t}}(K[\Delta])$ where $K[\Delta]$ is the Stanley-Reisner ring of a simplicial complex Δ , and studies the cohomology groups of this complex, termed the *enriched cohomology modules* of the simplicial complex. Fourthly, also in the square free case, K. Yanagawa, [15], shows that the third iteration $(A_{\mathbf{t}} \circ D_{\mathbf{t}})^3$ is isomorphic to the translation functor T^{-2n} on the derived category of square free modules.

In this paper we study the iterates of the Nakayama functor $\mathcal{N}_{\mathbf{t}} = A_{\mathbf{t}} \circ D_{\mathbf{t}}$ applied to a monomial quotient ring S/I where I is a positively \mathbf{t} -determined ideal, meaning that every minimal generator $x^{\mathbf{a}}$ of I has $\mathbf{a} \leq \mathbf{t}$. We compute the multigraded pieces of the cohomology groups of every iterate $\mathcal{N}_{\mathbf{t}}^k(S/I)$, Theorem 4.4. They turn out to be given by the reduced cohomology groups of various simplicial complexes derived from the monomial ideal I . In the square free case $\mathbf{t} = \mathbf{1}$ and $k = 1$ this specializes to Hochster's classical computation, [7], of the local cohomology modules of Stanley-Reisner rings, and in the case $k = 1$ and general \mathbf{t} we recover the more recent computations of Takayama, [11], of the local cohomology modules of a monomial quotient ring.

We consider the multigraded Betti spaces of the complex $\mathcal{N}_{\mathbf{t}}^k(S/I)$, and compute all these spaces for every k , Theorem 4.6. Again they are given by the reduced cohomology of various simplicial complexes derived from the monomial ideal I . In the case $k = 0$ we recover the well-known computations of the multigraded Betti spaces of S/I , see [9, Cor.5.12]. A striking feature is how similar in form the statements of Theorems 4.4 and 4.6 are, which compute respectively the multigraded cohomology and Betti spaces.

Not only are we able to compute the multigraded parts of the cohomology modules of $\mathcal{N}_{\mathbf{t}}^k(S/I)$ for every k . We also compute their S -module structure, Theorem 4.10, extending results of Gräbe, [5]. To give their S -module structure it is sufficient to show how multiplication with x_i acts on the various multigraded parts of the cohomology modules, and we show how these actions correspond to natural maps on reduced cohomology groups of simplicial complexes.

Actually all these computations are done in a more general setting. Since the interval $[0, \mathbf{t}]$ is a product $\prod_{i=1}^n [0, t_i]$, it turns out that for every \mathbf{k} in \mathbb{N}^n we may define the "multi-iterated" Nakayama functor $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}$. All our computations above are done, with virtually no extra effort, for the complexes $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ instead of for the simple iterates $\mathcal{N}_{\mathbf{t}}^k(S/I)$. Letting $\mathbf{t} + \mathbf{2}$ be $(t_1 + 2, t_2 + 2, \dots, t_n + 2)$ we also show that the multi-iteration $\mathcal{N}_{\mathbf{t}}^{\mathbf{t}+\mathbf{2}}$ is quasi-isomorphic to the translation functor T^{-2n} on complexes of positively \mathbf{t} -determined S -modules, thereby generalizing Yanagawa's result [15].

The organization of the paper is as follows. In Section 2 we recall the notion of Miller of positively \mathbf{t} -determined modules. We define the Nakayama functor $\mathcal{N}_{\mathbf{t}}$ on chain complexes of positively \mathbf{t} -determined modules, not as $A_{\mathbf{t}} \circ D_{\mathbf{t}}$ as explained in the introduction, but rather in a more direct way adapting slightly a defining property of local cohomology. More generally we define the multi-iterated Nakayama functor $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}$ for any \mathbf{k} in \mathbb{N}^n . In Section 3 we show that the Nakayama functor $\mathcal{N}_{\mathbf{t}}$ is the composition $A_{\mathbf{t}} \circ D_{\mathbf{t}}$ of Alexander duality and (twisted) standard duality. We also show how the category of positively \mathbf{t} -determined modules is equivalent to the category of modules over the incidence algebra of the poset $[0, \mathbf{t}]$, and that the composition $A_{\mathbf{t}} \circ D_{\mathbf{t}}$ corresponds to the Auslander-Reiten translate on the derived category of this module category. Section 4 gives our main results. This is the computation of the multigraded cohomology spaces and Betti spaces of $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ for a positively \mathbf{t} -determined ideal, and also the S -module structure of its cohomology groups.

In Section 5 we give sufficient conditions for vanishing of various cohomology groups of $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$. In particular we consider in detail the case when S is a polynomial ring in two variables, and investigate when $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ has only one non-vanishing cohomology module. In Section 6 we consider the Nakayama functor applied to quotient rings of S by ideals generated by variable powers. The essential thing is doing the case of one variable $n = 1$, and here we develop lemmata which will be pivotal in the final proofs. Section 7 considers posets. Two different constructions one may derive from a poset P are, algebraically, the KP -modules over the poset, and, topologically, the order complexes of various subposets of P . We give some results relating various Ext-groups of KP -modules, to the reduced cohomology groups of various associated order complexes.

A monomial ideal I will be positively \mathbf{t} -determined for many \mathbf{t} . In Section 8 we show how the complexes $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ for various \mathbf{t} (and \mathbf{k} depending on \mathbf{t}) are equivalent. Therefore various conditions like vanishing of cohomology modules and linearity conditions turn out to be independent of \mathbf{t} and thus intrinsic to the monomial ideal. Finally Section 9 contains the proofs of the main theorems given in Section 4.

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algebras, and that this functor in the setting of derived categories corresponds to the Auslander-Reiten translate.

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2. THE NAKAYAMA FUNCTOR AND LOCAL COHOMOLOGY

In this section we introduce the Nakayama functor and relate it to local cohomology. For a given \mathbf{t} in \mathbb{N}^n , Miller [8] has introduced the category of positively \mathbf{t} -determined S -modules, a subcategory of \mathbb{N}^n -graded modules. First we recall this module category. The Nakayama functor is then a functor applied to chain complexes of positively \mathbf{t} -determined modules.

2.1. Positively \mathbf{t} -determined modules. In this paper K denotes a fixed arbitrary field and S denotes the polynomial algebra $S = K[x_1, \dots, x_n]$ for $n \geq 1$. If $n = 1$ we write $S = K[x]$. Given $\mathbf{a} = (a_1, \dots, a_n)$ in the monoid \mathbb{N}^n , we shall use the notation

$$x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n} \quad \text{and} \quad \mathbf{m}^{\mathbf{a}} = (x_i^{a_i} : a_i \geq 1).$$

Let $\mathbf{t} \in \mathbb{N}^n$. We shall begin this section by recalling Miller's concept of positively \mathbf{t} -determined ideals in S and how this concept extends to \mathbb{N}^n -graded S -modules. We use the partial order on \mathbb{N}^n where $\mathbf{a} \leq \mathbf{b}$ if each coordinate $a_i \leq b_i$. Similarly we have a partial order on \mathbb{Z}^n .

First of all a monomial ideal I in S is positively \mathbf{t} -determined if every minimal generator $x^{\mathbf{a}}$ of I satisfies $\mathbf{a} \leq \mathbf{t}$. In particular, with the notation $\mathbf{1} = (1, \dots, 1)$ the ideal I is $\mathbf{1}$ -determined if and only if it is monomial and square free. Miller has extended this concept to \mathbb{N}^n -graded S -modules. Let $\varepsilon_i \in \mathbb{N}^n$ be the i 'th unit vector with 1 in position i and zero in other positions. An \mathbb{N}^n -graded module M is *positively \mathbf{t} -determined* if the multiplication

$$M_{\mathbf{a}} \xrightarrow{\cdot x_i} M_{\mathbf{a} + \varepsilon_i}$$

is an isomorphism whenever $a_i \geq t_i$. Note that a monomial ideal I in S is positively \mathbf{t} -determined if and only if I considered as a \mathbb{Z}^n -graded S -module is positively \mathbf{t} -determined. This again holds if and only if S/I is positively \mathbf{t} -determined.

Let us agree that an \mathbb{N}^n -graded module is also \mathbb{Z}^n -graded by letting it be zero in all degrees in $\mathbb{Z}^n \setminus \mathbb{N}^n$. Conversely a \mathbb{Z}^n -graded module is called \mathbb{N}^n -graded if it is zero in all degrees in $\mathbb{Z}^n \setminus \mathbb{N}^n$.

Given an order preserving map $q: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ and a \mathbb{Z}^n -graded S -module M . We then get a \mathbb{Z}^n -graded S -module q^*M with $(q^*M)_{\mathbf{a}} = M_{q(\mathbf{a})}$, and with multiplication $x^{\mathbf{b}}: (q^*M)_{\mathbf{a}} \rightarrow (q^*M)_{\mathbf{a}+\mathbf{b}}$ defined to be multiplication with $x^{q(\mathbf{a}+\mathbf{b})-q(\mathbf{a})}$ from $M_{q(\mathbf{a})}$ to $M_{q(\mathbf{a}+\mathbf{b})}$.

Example 2.1. Given a \mathbb{Z}^n -graded S -module M and $\mathbf{a} \in \mathbb{N}^n$ there is a \mathbb{Z}^n -graded S -module $M(\mathbf{a})$ with $M(\mathbf{a})_{\mathbf{r}} = M_{\mathbf{a}+\mathbf{r}}$. In the above terminology $M(\mathbf{a}) = \tau_{\mathbf{a}}^*M$ for the order preserving translation map $\tau_{\mathbf{a}}: \mathbb{Z}_*^n \rightarrow \mathbb{Z}_*^n$ with $\tau_{\mathbf{a}}(\mathbf{r}) = \mathbf{a} + \mathbf{r}$.

A variation of this, actually this is the only situation we shall be concerned with, is when we have an order preserving map $q: \mathbb{N}^n \rightarrow \mathbb{Z}^n$. Then q^*M is \mathbb{N}^n graded (and hence by our conventions \mathbb{Z}^n -graded by considering it to be zero in all degrees in $\mathbb{Z}^n \setminus \mathbb{N}^n$).

To a \mathbb{Z}^n -graded S -module and a multidegree \mathbf{t} in \mathbb{N}^n we can associate a positively \mathbf{t} -determined module as follows.

Definition 2.2. Let $p_{\mathbf{t}}: \mathbb{N}^n \rightarrow \mathbb{Z}^n$ be given as $p_{\mathbf{t}}(\mathbf{a}) = (p_{t_1}(a_1), \dots, p_{t_n}(a_n))$ where the coordinates are defined by

$$p_{t_i}(a_i) = \begin{cases} a_i & \text{if } 0 \leq a_i \leq t_i, \\ t_i & \text{if } a_i \geq t_i. \end{cases}$$

The \mathbf{t} -truncation of M is then $p_{\mathbf{t}}^*M$.

Note that the \mathbf{t} -truncation of M only depends on the module structure of M for degrees in the interval $[0, \mathbf{t}]$.

Example 2.3. Let $n = 1$. Given $r \geq 1$ and $t \geq 0$ we have:

$$p_t^*(S/x^r S) = \begin{cases} S & \text{if } t \leq r - 1 \\ S/x^r S & \text{if } r \leq t. \end{cases}$$

On the other hand

$$p_t^*(x^r S) = \begin{cases} 0 & \text{if } t \leq r - 1 \\ x^r S & \text{if } r \leq t. \end{cases}$$

The above example show that homologically the functor $M \mapsto p_{\mathbf{t}}^*M$ is somewhat ill-behaved because it takes some modules of rank zero to modules of rank one and some modules of rank one to zero. However in this paper we take advantage of this, much like we usually take advantage of non-exactness to obtain derived functors.

Note that if $f: M \rightarrow N$ is a homomorphism of \mathbb{Z}^n -graded S -modules, and q is an order preserving map from \mathbb{N}^n or \mathbb{Z}^n , to \mathbb{Z}^n , then the homomorphisms $f_{q(\mathbf{a})}: M_{q(\mathbf{a})} \rightarrow N_{q(\mathbf{a})}$ induce a homomorphism $q^*f: q^*M \rightarrow q^*N$ of \mathbb{Z}^n -graded S -modules. Thus $M \mapsto q^*M$ is an endofunctor on the category of \mathbb{Z}^n -graded S -modules. This endofunctor is exact in the following sense.

Lemma 2.4. Let $L \xrightarrow{g} M \xrightarrow{f} N$ be an exact sequence of \mathbb{Z}^n -graded S -modules and q an order preserving map from \mathbb{N}^n or \mathbb{Z}^n , to \mathbb{Z}^n . The induced sequence $q^*L \xrightarrow{q^*g} q^*M \xrightarrow{q^*f} q^*N$ is exact.

For $\mathbf{t} \in \mathbb{N}^n$ every positively \mathbf{t} -determined S -module has a filtration of modules which are isomorphic to positively \mathbf{t} -determined S -modules of the following form.

Definition 2.5. Given \mathbf{t} , \mathbf{a} and \mathbf{b} in \mathbb{N}^n with $\mathbf{b} \leq \mathbf{t}$ and $\mathbf{a} \leq \mathbf{b}$. The positively \mathbf{t} -determined interval S -module $K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\}$ is defined as:

$$K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\} = p_{\mathbf{t}}^*((S/\mathfrak{m}^{\mathbf{b}-\mathbf{a}+1})(-\mathbf{a}))$$

Note that for all \mathbf{r} in the interval $[0, \mathbf{t}]$ the graded piece $K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\}_{\mathbf{r}}$ is K if $\mathbf{r} \in [\mathbf{a}, \mathbf{b}]$ and is zero if $\mathbf{r} \notin [\mathbf{a}, \mathbf{b}]$. It will also be convenient to have the convention that $K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\}$ is zero if \mathbf{a} is not less or equal to \mathbf{b} .

Example 2.6. Let us consider the case $n = 1$. Given $t \in \mathbb{N}$ and $0 \leq a \leq b \leq t$ we have

$$K_t\{a, b\} = \begin{cases} (S/x^{b-a+1}S)(-a) & \text{if } b < t \\ S(-a) & \text{if } b = t. \end{cases}$$

Note that if $a \leq a' \leq b$, there is an inclusion $i: K_t\{a', b\} \rightarrow K_t\{a, b\}$, and that if $a \leq b \leq b'$ there is a surjection $p: K_t\{a, b'\} \rightarrow K_t\{a, b\}$. We shall call i the canonical inclusion and p the canonical projection.

Taking tensor products over K we have the isomorphism

$$K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\} \cong \bigotimes_{j=1}^n K_{t_j}\{a_j, b_j\}$$

of modules over $S = \bigotimes_{j=1}^n K[x_j]$.

2.2. Local cohomology. We let $\underline{\text{Ext}}$ denote the \mathbb{Z}^n -graded version of the Ext-functor. The local cohomology modules of a \mathbb{Z}^n -graded S -module M are the \mathbb{Z}^n -graded S -modules

$$H_{\mathbf{m}}^i(M) = \text{colim}_k \underline{\text{Ext}}_S^i(S/(\mathbf{m}^k), M),$$

where $\mathbf{m} = (x_1, \dots, x_n)$ is the unique monomial maximal ideal in S . Instead of the modules \mathbf{m}^k any sequence of modules containing these as a cofinal sub-sequence may be used. In particular this local cohomology module is the colimit of $\underline{\text{Ext}}_S^i(S/\mathbf{m}^{\mathbf{t}}, M)$ as \mathbf{t} tends to infinity. Given $t \in \mathbb{N}$ we let $P_{t\epsilon_j}$ denote the projective resolution of $S/x_j^t S$ given by the inclusion $x_j^t S \rightarrow S$. We will work with chain complexes throughout, that is, differentials decrease the homological degree. If C_* is a chain complex we shall write $H^i C$ for $H_{-i} C$. The Ext-group $\underline{\text{Ext}}_S^i(S/\mathbf{m}^{\mathbf{t}}, M)$ is the i 'th cohomology module of the chain complex $\underline{\text{Hom}}_S(P_{\mathbf{t}}, M)$ where $P_{\mathbf{t}}$, given by the tensor product $P_{\mathbf{t}} = \bigotimes_{j=1}^n P_{t_j\epsilon_j}$ over S , is the Koszul complex for $S/\mathbf{m}^{\mathbf{t}}$ and $\underline{\text{Hom}}_S$ denotes the \mathbb{Z}^n -graded version of Hom_S .

Proposition 2.7. *Let $\mathbf{t} \in \mathbb{N}^n$ and let M be a positively \mathbf{t} -determined S -module. If $\mathbf{t}' \geq \mathbf{t}$ then the homomorphism*

$$p_{\mathbf{t}}^* \underline{\text{Ext}}_S^i(S/\mathbf{m}^{\mathbf{t}+1}, M(-1)) \rightarrow p_{\mathbf{t}'}^* \underline{\text{Ext}}_S^i(S/\mathbf{m}^{\mathbf{t}'+1}, M(-1))$$

is an isomorphism for all i .

Proof. The homomorphism $P_{\mathbf{t}'+1} \rightarrow P_{\mathbf{t}+1}$ induces an isomorphism

$$p_{\mathbf{t}}^* \underline{\text{Hom}}_S(P_{\mathbf{t}+1}, M(-1)) \rightarrow p_{\mathbf{t}'}^* \underline{\text{Hom}}_S(P_{\mathbf{t}'+1}, M(-1)).$$

In order to see this it suffices to consider the case $n = 1$ and to show that the homomorphism

$$M(t) \cong \underline{\text{Hom}}_S(x^{t+1} S, M(-1)) \rightarrow \underline{\text{Hom}}_S(x^{t'+1} S, M(-1)) \cong M(t')$$

induces an isomorphism $p_t^* M(t) \rightarrow p_{t'}^* M(t')$. This is the case because since M is t -determined the multiplication by $x^{t'-t}$ induces an isomorphism

$$M(t)_a = M_{t+a} \xrightarrow{x^{t'-t}} M_{t'+a} = M(t')_a$$

for all $a \geq 0$. □

The above proposition shows that if M is positively \mathbf{t} -determined then the \mathbf{t} -truncation of $H_{\mathbf{m}}^i(M(-1))$ is isomorphic to the \mathbf{t} -truncation of $\underline{\text{Ext}}_S^i(S/\mathbf{m}^{\mathbf{t}+1}, M(-1))$. The following proposition shows that in this case the \mathbb{Z}^n -graded S -module $H_{\mathbf{m}}^i(M(-1))$ is completely determined by its \mathbf{t} -truncation.

Proposition 2.8. *Let $\mathbf{t} \in \mathbb{N}^n$ and let M be a positively \mathbf{t} -determined S -module.*

(1) *If $a_j \geq t_j + 1$ for some j then $H_{\mathbf{m}}^i(M(-1))_{\mathbf{a}}$ is zero.*

(2) If $a_j \leq -1$ for some j then the multiplication map

$$H_{\mathfrak{m}}^i(M(-\mathbf{1}))_{\mathbf{a}} \xrightarrow{x_j} H_{\mathfrak{m}}^i(M(-\mathbf{1}))_{\mathbf{a}+\varepsilon_j}$$

is an isomorphism.

Proof. Since the statements only concern the j 'th graded direction we can use the adjunction between \otimes and $\underline{\text{Hom}}$ to reduce to the case $n = 1$. For (1) we note that if M is t -determined, then the homomorphism

$$M_{a-1} \cong \underline{\text{Hom}}_S(S, M(-1))_a \rightarrow \underline{\text{Hom}}_S(x^r S, M(-1))_a \cong M_{a-1+r},$$

induced by multiplication by x^r , is an isomorphism whenever $a \geq t + 1$. Thus the group $\text{Ext}_S^i(S/\mathfrak{m}^r, M(-1))_a$ is zero for all r and i . This implies that the degree a part of local cohomology of M is zero.

For (2) we consider the resolution $P_r = (x^r S \rightarrow S)$ of $S/x^r S$. We shall show that for $a \leq -1$ the multiplication

$$x: \underline{\text{Hom}}_S(P_r, M(-1))_a \rightarrow \underline{\text{Hom}}_S(P_r, M(-1))_{a+1}$$

is an isomorphism for all sufficiently large r . Since $M_b = 0$ for $b < 0$ the zero cochains of both $\underline{\text{Hom}}_S(P_r, M(-1))_a$ and $\underline{\text{Hom}}_S(P_r, M(-1))_{a+1}$ are zero. Thus we only need to consider one cochains, that is, the multiplication

$$x_j: M_{r+a-1} \rightarrow M_{r+a}.$$

Since M is t -determined this is an isomorphism provided that $r + a \geq t + 1$. \square

Remark 2.9. Using the terminology of [8], Section 2, this may be phrased as saying that the local cohomology of $M(-\mathbf{1})$ is negatively \mathbf{t} -determined. This may be demonstrated by providing $\mathbf{t} + \mathbf{1}$ -determined injective resolutions of $M(-\mathbf{1})$. We thank E. Miller for pointing this out.

2.3. The Nakayama Functor. We now shift attention from the local cohomology of a \mathbb{Z}^n -graded S -module M to the groups $\underline{\text{Ext}}_S^i(S/\mathfrak{m}^{\mathbf{t}+\mathbf{1}}, M(-\mathbf{1}))$ and the chain complexes that compute them. We construct an endofunctor $M \mapsto \mathcal{N}_{\mathbf{t}}^1(M)$, called the Nakayama functor, on the category of chain complexes of positively \mathbf{t} -determined S -modules. The Nakayama functor is designed so that for every $\mathbf{t} \geq 0$ there is an isomorphism

$$H^i \mathcal{N}_{\mathbf{t}}^1(M) \cong p_{\mathbf{t}}^* \underline{\text{Ext}}_S^i(S/\mathfrak{m}^{\mathbf{t}+\mathbf{1}}, M(-\mathbf{1}))$$

of \mathbb{Z}^n -graded S -modules. In view of Propositions 2.7 and 2.8 the local cohomology of a positively \mathbf{t} -determined S -module M is completely determined by the cohomology of $\mathcal{N}_{\mathbf{t}}^1(M)$, and vice versa. In fact $H^i \mathcal{N}_{\mathbf{t}}^1(M)$ will be $p_{\mathbf{t}}^*$ of the local cohomology module $H_{\mathfrak{m}}^i M(-\mathbf{1})$.

Recall that for $t \geq 0$ we denote the resolution $x_j^t S \rightarrow S$ of $S/x_j^t S$ by $P_{t\varepsilon_j}$. Given $\mathbf{k} \in \{0, 1\}^n$ we let $P_{\mathbf{t}}^{\mathbf{k}}$ denote the tensor product $P_{\mathbf{t}}^{\mathbf{k}} = \bigotimes_{j=1}^n P_{t_j\varepsilon_j}^{\otimes k_j}$ with the convention that undecorated tensor products are over S and that the zeroth tensor power of an S -module is S itself.

Definition 2.10. Given $\mathbf{t} \in \mathbb{N}^n$ and $\mathbf{k} \in \{0, 1\}^n$ the \mathbf{k} -iterate of the \mathbf{t} -Nakayama functor is the endofunctor on the category of chain complexes of positively \mathbf{t} -determined S -modules given by the formula

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(C) = p_{\mathbf{t}}^* \underline{\text{Hom}}_S(P_{\mathbf{t}+\mathbf{1}}^{\mathbf{k}}, C(-\mathbf{k})).$$

We shall shortly generalize this so that we for any $\mathbf{k} \in \mathbb{N}^n$ define a functor $C \mapsto \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(C)$. Our goal is to compute the cohomology of $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ for a positively \mathbf{t} -determined monomial ideal I . These computations will eventually reduce to the case when $n = 1$ and the module is an interval module, as in the following example.

Remark 2.11. As said above, when $\mathbf{k} = \mathbf{1}$, the cohomology of $\mathcal{N}_{\mathbf{t}}^1(M)$ for a module M is essentially the local cohomology of $M(-1)$. Theorem 6.2 of [8] or rather its proof computes the local cohomology module in essentially the same way as in the above definition.

Example 2.12. Let $n = 1$ and $0 \leq a \leq b \leq t$. There are isomorphisms

$$\underline{\mathrm{Hom}}_S(x^{t+1}S, K_t\{a, b\}(-1)) \cong K_t\{a, b\}(t)$$

and

$$\underline{\mathrm{Hom}}_S(S, K_t\{a, b\}(-1)) \cong K_t\{a, b\}(-1)$$

Taking the \mathbf{t} -truncation, it follows that if $b < t$ then $\mathcal{N}_t^1(K_t\{a, b\})$ is isomorphic to the chain complex

$$K_t\{a+1, b+1\} \rightarrow 0,$$

and if $b = t$ then $\mathcal{N}_t^1(K_t\{a, b\})$ is isomorphic to the chain complex

$$K_t\{a+1, t\} \rightarrow K_t\{0, t\}.$$

On cohomology we obtain

$$H^*\mathcal{N}_t^1 K_t\{a, b\} \cong \begin{cases} K_t\{a+1, b+1\}e^0 & \text{if } b < t \\ K_t\{0, a\}e^1 & \text{if } b = t. \end{cases}$$

Here e^0 is a generator in cohomological degree zero and e^1 is a generator in cohomological degree one.

Note that if $\mathbf{k}, \mathbf{k}' \in \{0, 1\}^n$ are such that $\mathbf{k} + \mathbf{k}' \in \{0, 1\}^n$, then the adjunction between $\underline{\mathrm{Hom}}$ and the tensor product over S induces an isomorphism

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}'}(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(C)) \cong \mathcal{N}_{\mathbf{t}}^{\mathbf{k}'+\mathbf{k}}(C).$$

We now define $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}$ in general.

Definition 2.13. Given multidegrees $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{t} = (t_1, \dots, t_n)$ in \mathbb{N}^n , we define the \mathbf{k} -iteration of the \mathbf{t} -Nakayama functor on C , denoted $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(C)$, to be the chain complex

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(C) = (\mathcal{N}_{t_1\varepsilon_1}^{\varepsilon_1})^{\circ k_1} \circ \dots \circ (\mathcal{N}_{t_n\varepsilon_n}^{\varepsilon_n})^{\circ k_n}(C).$$

This gives an endofunctor $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}$ on the category of chain complexes of positively \mathbf{t} -determined S -modules.

Remark 2.14. In this paper we work in the category of chain complexes, and not in the derived category. The reason is that we want to work with explicit morphisms between chain complexes. The only place we refer to the derived category is in Section 3 where we relate the Nakayama functor to the Auslander-Reiten translate.

Remark 2.15. Two facts we will use on some occasions are the following. Since $P_{\mathbf{t}+\mathbf{1}}^{\mathbf{k}}$ in Definition 2.10 is a bounded complex of finitely generated projectives, the Nakayama functor commutes with colimits. Also, since the functor $\underline{\mathrm{Hom}}_S(P_{\mathbf{t}+\mathbf{1}}^{\mathbf{k}}, -)$ is a right adjoint it commutes with limits, and so also the Nakayama functor.

One might expect that the effect of the truncation $p_{\mathbf{t}}$ is minor. However this is not the case. Given a chain complex C we let TC denote the homologically shifted chain complex with $(TC)_i = C_{i-1}$ and with $d_{TC} = -d_C$. In Section 6 we show that the truncation $p_{\mathbf{t}}$ gives the Nakayama functor $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}$ the following remarkable property, generalizing a result of Yanagawa [15] in the case $\mathbf{t} = \mathbf{1}$.

Proposition 2.16. *Let $\mathbf{t} \in \mathbb{N}^n$ and let C be a chain complex of positively \mathbf{t} -determined S -modules. For every j the evaluation $\mathcal{N}_{\mathbf{t}}^{(t_j+2)\varepsilon_j} C$ of the Nakayama functor at C is naturally quasi-isomorphic to $T^{-2}C$. In particular, for every \mathbf{k} in \mathbb{N}^n the Nakayama functor $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}$ induces a self-equivalence on the derived category of the category of chain complexes of positively \mathbf{t} -determined S -modules.*

3. DUALITY FUNCTORS

In this section we describe the Nakayama functor as the composition of two duality functors. The Nakayama functor in the theory of Artin algebras is defined on the category of chain complexes of left modules over an Artin algebra. It is the composition of two duality functors and it represents the Auslander–Reiten translate on the bounded derived category of chain complexes of finitely generated positively \mathbf{t} -determined modules. Let us denote the incidence algebra of the partially ordered set P consisting of the multidegrees $\mathbf{a} \in \mathbb{Z}^n$ satisfying $0 \leq \mathbf{a} \leq \mathbf{t}$ by $\Lambda_{\mathbf{t}}$. We shall show that under an equivalence of categories between the category of \mathbf{t} -determined S -modules and the category of left $\Lambda_{\mathbf{t}}$ -modules the functor $\mathcal{N}_{\mathbf{t}}^{\mathbf{1}}$ corresponds to the Nakayama functor from the theory of Artin algebras.

This section is not a prerequisite for the rest of the paper. The reader who wishes may safely go directly to Section 4 where the main results are presented.

Note. Our definition of the Nakayama functor, Definition 2.13, works well whether M is finitely generated or not. However since we here compare it to the composition of two duality functors, we assume in this section that all modules have graded parts which are finite-dimensional vector spaces, in particular all positively \mathbf{t} -determined modules are finitely generated.

3.1. Local duality. There are two highly different duals of a \mathbb{Z}^n -graded S -module M . One is the \mathbb{Z}^n -graded Matlis dual $A(M) = \underline{\mathrm{Hom}}_S(M, E)$, where E is the \mathbb{Z}^n -graded injective envelope $E = K[x_1^{-1}, \dots, x_n^{-1}]$ of S/\mathfrak{m} . The other is the chain complex $D(M) = \underline{\mathrm{Hom}}_S(M, Q)$ where Q is an injective resolution of $T^n S(-\mathbf{1})$. Here T^n shifts homological degrees up by n . The complex Q is a dualizing complex for finitely generated \mathbb{Z}^n -graded S -modules in the sense that the natural homomorphism $M \rightarrow (D \circ D)(M)$ is a quasi-isomorphism for every finitely generated \mathbb{Z}^n -graded S -module M . Grothendieck’s local duality theorem provides an isomorphism of \mathbb{Z}^n -graded S -modules between $H^i(A \circ D(M))$ and the local cohomology $H_{\mathfrak{m}}^i(M)$ [2, Example 3.6.10 and Theorem 3.6.19]. Below we study twisted versions $M \mapsto p_{\mathbf{t}}^* A(M(\mathbf{t}))$ and $M \mapsto D(M(\mathbf{t} - \mathbf{1}))$ of the duality functors A and D .

3.2. Alexander duality. In [8] Miller introduced the Alexander duality functor on the category of positively \mathbf{t} -determined S -modules. The value on a positively \mathbf{t} -determined S -module M is given by the formula

$$A_{\mathbf{t}}(M) = p_{\mathbf{t}}^* A(M(\mathbf{t})) = p_{\mathbf{t}}^* \underline{\mathrm{Hom}}_S(M(\mathbf{t}), E).$$

The functor $A_{\mathbf{t}}$ generalizes Alexander duality for simplicial complexes.

In order to make the definition of $A_{\mathbf{t}}(M)$ more explicit we note that if $\mathbf{a} \in [0, \mathbf{t}]$, then there is an isomorphism

$$A_{\mathbf{t}}(M)_{\mathbf{a}} \cong \underline{\mathrm{Hom}}_K(M(\mathbf{t}), K)_{\mathbf{a}} = \mathrm{Hom}_K(M_{\mathbf{t}-\mathbf{a}}, K).$$

If $\mathbf{a} + \varepsilon_i \leq \mathbf{t}$, then the multiplication map

$$x_i: A_{\mathbf{t}}(M)_{\mathbf{a}} \rightarrow A_{\mathbf{t}}(M)_{\mathbf{a}+\varepsilon_i}$$

is dual to the multiplication

$$x_i: M_{\mathbf{t}-\mathbf{a}-\varepsilon_i} \rightarrow M_{\mathbf{t}-\mathbf{a}}.$$

If M is positively \mathbf{t} -determined, then $A_{\mathbf{t}} \circ A_{\mathbf{t}}(M)$ is isomorphic to M .

Example 3.1. Let I be a monomial ideal in S and suppose that the minimal generators in degrees less than or equal to \mathbf{t} are given by the monomials $x^{\mathbf{a}^1}, \dots, x^{\mathbf{a}^r}$. Following Miller and Sturmfels [9] we let $I^{[\mathbf{t}]}$ denote the Alexander dual positively \mathbf{t} -determined ideal

$$I^{[\mathbf{t}]} = \mathfrak{m}^{\mathbf{b}^1} \cap \dots \cap \mathfrak{m}^{\mathbf{b}^r},$$

where

$$b_j^i = \begin{cases} t_j + 1 - a_j^i & \text{if } a_j^i \geq 1 \\ 0 & \text{if } a_j^i = 0. \end{cases}$$

The \mathbb{Z}^n -graded S -modules $A_{\mathbf{t}}(I)$ and $S/I^{[\mathbf{t}]}$ are isomorphic.

Example 3.2. Let $n = 1$ and let $M = K_t\{a, b\}$ for $0 \leq a \leq b \leq t$. There is an isomorphism $A_t(M) \cong K_t\{t-b, t-a\}$. In particular

$$A_t(K_t\{a, t\}) = A_t(S(-a)) = \begin{cases} S & \text{if } a = 0 \\ S/x^{t-a+1}S & \text{if } a > 0. \end{cases}$$

3.3. A twisted duality. We shall now study the twisted duality

$$M \mapsto D(M(\mathbf{t} - \mathbf{1})) = \underline{\mathrm{Hom}}_S(M, Q(\mathbf{1} - \mathbf{t})).$$

It has the property that $p_{\mathbf{t}}^* \circ A \circ D(M(-\mathbf{1}))$ is equal to $A_{\mathbf{t}} \circ D(M(\mathbf{t} - \mathbf{1}))$. Since we are interested in the Nakayama functor $\mathcal{N}_{\mathbf{t}}^1(M)$ on a positively \mathbf{t} -determined S -module M , it will be convenient to work with a version $M \mapsto D_{\mathbf{t}}(M)$ of this twisted duality functor which restricts to an endofunctor on the category of positively \mathbf{t} -determined S -modules.

For $\mathbf{t} \in \mathbb{N}^n$ recall the complex $P_{\mathbf{t}} = P_{\mathbf{t}}^1$ of Subsection 2.3. We let $I_{\mathbf{t}}$ denote the injective resolution

$$I_{\mathbf{t}} = \underline{\mathrm{Hom}}_S(\underline{\mathrm{Hom}}_S(P_{\mathbf{t}}, S), K[x_1^{-1}, \dots, x_n^{-1}]) \cong \underline{\mathrm{Hom}}_K(\underline{\mathrm{Hom}}_S(P_{\mathbf{t}}, S), K)$$

of $T^n S/\mathfrak{m}^{\mathbf{t}}(-\mathbf{1})$. We define $D_{\mathbf{t}}$ to be the endofunctor on the category of chain complexes of \mathbb{Z}^n -graded S -modules given by the formula

$$D_{\mathbf{t}}(C) = p_{\mathbf{t}}^* \underline{\mathrm{Hom}}_S(C, I_{\mathbf{t}+\mathbf{1}}(\mathbf{1} - \mathbf{t})).$$

We shall show in Proposition 3.6 that this is essentially the same as the functor $D(M(\mathbf{t} - \mathbf{1}))$.

Example 3.3. Let $n = 1$. Then I_{t+1} is isomorphic to the chain complex $K[x^{-1}](-t-1) \rightarrow K[x^{-1}]$ with $K[x^{-1}](-t-1)$ in degree one. Let $0 \leq a \leq b \leq t$. There are isomorphisms

$$\underline{\mathrm{Hom}}_S(K_t\{a, b\}, K[x^{-1}](-t-1))(1-t) \cong \underline{\mathrm{Hom}}_K(K_t\{a, b\}, K)(-2t)$$

and

$$\underline{\mathrm{Hom}}_S(K_t\{a, b\}, K[x^{-1}])(1-t) \cong \underline{\mathrm{Hom}}_K(K_t\{a, b\}, K)(1-t).$$

If $b < t$ then $D_t(K_t\{a, b\})$ is isomorphic to the chain complex

$$0 \rightarrow K_t\{t-b-1, t-a-1\}$$

and if $b = t$, it is isomorphic to the chain complex

$$K_t\{0, t\} \rightarrow K_t\{0, t-a-1\}.$$

On homology we obtain an isomorphism

$$H_*D_t(K_t\{a, b\}) \cong \begin{cases} K_t\{t-b-1, t-a-1\}e_0 & \text{if } b < t \\ K_t\{t-a, t\}e_1 & \text{if } b = t. \end{cases}$$

Here e_0 is a generator in homological degree zero and e_1 is a generator in homological degree one.

Lemma 3.4. *Let M be a finitely generated \mathbb{Z}^n -graded S -module. For every $\mathbf{r} \in \mathbb{N}^n$ there is a natural isomorphism*

$$\underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, M) \cong \underline{\mathrm{Hom}}_K(\underline{\mathrm{Hom}}_S(M, I_{\mathbf{r}}), K)$$

Proof. First note that $M \otimes_S \underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, S)$ is naturally isomorphic to $\underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, M)$. Together with the adjunction isomorphism

$$\underline{\mathrm{Hom}}_S(M, \underline{\mathrm{Hom}}_K(\underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, S), K)) \cong \underline{\mathrm{Hom}}_K(M \otimes_S \underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, S), K)$$

this gives an isomorphism $\underline{\mathrm{Hom}}_S(M, I_{\mathbf{r}}) \cong \underline{\mathrm{Hom}}_K(\underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, M), K)$. The result now follows from the isomorphism $N \cong \underline{\mathrm{Hom}}_K(\underline{\mathrm{Hom}}_K(N, K), K)$ with $N = \underline{\mathrm{Hom}}_S(P_{\mathbf{r}}, M)$. \square

The following is the main observation we make in this subsection.

Corollary 3.5. *For every $\mathbf{t} \in \mathbb{N}^n$ there is an isomorphism*

$$\mathcal{N}_{\mathbf{t}}^1 \cong A_{\mathbf{t}} \circ D_{\mathbf{t}}$$

of endofunctors on the category of chain complexes of finitely generated positively \mathbf{t} -determined S -modules.

Proof. Applying the truncation $p_{\mathbf{t}}$ and Lemma 3.4 we obtain isomorphisms

$$\begin{aligned} p_{\mathbf{t}}^* \underline{\mathrm{Hom}}_S(P_{\mathbf{t}+1}, C(-1)) &\cong A_{\mathbf{t}}(\underline{\mathrm{Hom}}_S(C, I_{\mathbf{t}+1}(\mathbf{1}-\mathbf{t}))) \\ &\cong A_{\mathbf{t}}(p_{\mathbf{t}}^* \underline{\mathrm{Hom}}_S(C, I_{\mathbf{t}+1}(\mathbf{1}-\mathbf{t}))) = A_{\mathbf{t}} \circ D_{\mathbf{t}}(C). \end{aligned}$$

\square

The following shows that our duality $D_{\mathbf{t}}$ is a version for positively \mathbf{t} -determined modules of the standard duality D . Also since $A_{\mathbf{t}}$ is exact, the above corollary together with the following proposition implies the local duality theorem for \mathbb{Z}^n -graded S -modules.

Proposition 3.6. *Let M be a positively \mathbf{t} -determined S -module. There is a natural quasi-isomorphism between the chain complexes $D_{\mathbf{t}}(M)$ and $D(M(\mathbf{t} - \mathbf{1}))$. In particular there is a natural isomorphism*

$$H_i D_{\mathbf{t}}(M) \cong \underline{\mathrm{Ext}}_S^{n-i}(M, S(-\mathbf{t})).$$

Proof. It suffices to show that if P is a projective positively \mathbf{t} -determined S -module, then there is a quasi-isomorphism $T^n \underline{\mathrm{Hom}}_S(P, S(-\mathbf{t})) \xrightarrow{\sim} D_{\mathbf{t}}(P)$ and that this quasi-isomorphism is natural in P . However since $I_{\mathbf{t}+1}$ is a resolution of $T^n S/\mathfrak{m}^{\mathbf{t}+1}(-\mathbf{1})$ and P is projective there is a quasi-isomorphism

$$T^n p_{\mathbf{t}*} [\underline{\mathrm{Hom}}_S(P, S/\mathfrak{m}^{\mathbf{t}+1}(-\mathbf{1}))(\mathbf{1} - \mathbf{t})] \rightarrow D_{\mathbf{t}}(P).$$

On the other hand every projective positively \mathbf{t} -determined S -module is a direct sum of S -modules of the form $S(-\mathbf{a})$ for $0 \leq \mathbf{a} \leq \mathbf{t}$ and we have isomorphisms

$$\begin{aligned} p_{\mathbf{t}}^* [\underline{\mathrm{Hom}}_S(S(-\mathbf{a}), S/\mathfrak{m}^{\mathbf{t}+1} S(-\mathbf{1}))(\mathbf{1} - \mathbf{t})] &\cong p_{\mathbf{t}}^* [(S/\mathfrak{m}^{\mathbf{t}+1} S)(\mathbf{a} - \mathbf{t})] \\ &\cong S(\mathbf{a} - \mathbf{t}) \\ &\cong \underline{\mathrm{Hom}}_S(S(-\mathbf{a}), S(-\mathbf{t})). \end{aligned}$$

□

3.4. Squarefree modules. In the case $\mathbf{t} = \mathbf{1}$ positively \mathbf{t} -determined S -modules are termed squarefree S -modules by Yanagawa [14]. Alexander duality for square free modules, defined by T.Römer [10] and E.Miller [8], has been well studied. For instance the Alexander dual of a Stanley-Reisner ring $K[\Delta]$ is the Stanley-Reisner ideal I_{Δ^*} of the Alexander dual simplicial complex Δ^* . In [4], the cohomology modules of $\mathcal{N}_1^1(S/I)$ were termed the enriched cohomology modules of Δ by the second author, because the S -module rank of $H^i \mathcal{N}_1^1(S/I)$ equals the K -vector space dimension of the reduced cohomology $\tilde{H}^i(\Delta)$. Yanagawa, [15], observed that in the derived category of chain complexes of positively $\mathbf{1}$ -determined S -modules, which he calls the category of squarefree S -modules, the three times iterated Nakayama functor $\mathcal{N}_1^1 \circ \mathcal{N}_1^1 \circ \mathcal{N}_1^1(M)$ is isomorphic to a homological shift of M [15]. This is an instance of Proposition 2.16.

3.5. Duality over incidence algebras. In order to give our next description of the Nakayama functor we note that up to canonical isomorphism a positively \mathbf{t} -determined S -module M is determined by the K -vector spaces $M_{\mathbf{a}}$ for $0 \leq \mathbf{a} \leq \mathbf{t}$ and the multiplication homomorphisms $x_j: M_{\mathbf{a}-\varepsilon_j} \rightarrow M_{\mathbf{a}}$ for $\varepsilon_j \leq \mathbf{a} \leq \mathbf{t}$. This is exactly the structure encoded by a (left) module over the incidence algebra $\Lambda_{\mathbf{t}}$ of the poset $[0, \mathbf{t}] = \{\mathbf{a}: 0 \leq \mathbf{a} \leq \mathbf{t}\}$.

Let us recall that to every finite poset P there is an associated incidence algebra, denoted $I(P)$. It is the K -algebra which as a K -vector space has the pairs (q, p) (where $p \leq q$) as basis. The product is given on basis elements by $(r, q) \cdot (q, p) = (r, p)$, and $(r, q') \cdot (q, p)$ is zero when $q \neq q'$. The identity element in the incidence algebra is $\sum_{p \in P} (p, p)$. A (left) module M over the incidence algebra may then be written as $M = \sum_{p \in P} (p, p) \cdot M$ or, letting $M_p = (p, p) \cdot M$, we get $M = \oplus_{p \in P} M_p$. The category of (left) modules over $I(P)$ is isomorphic to the category of functors from P , considered as a category with a morphism $p \rightarrow p'$ if and only if $p \leq p'$, to the category of K -vector spaces. Such functors are often called KP -modules.

Given two (left) $I(P)$ -modules M and N we denote by $\text{Hom}_{I(P)}(M, N)$ the K -vector space of $I(P)$ -linear homomorphisms from M to N . Since $I(P)$ is an $I(P)$ -bimodule, $\text{Hom}_{I(P)}(M, I(P))$ becomes a right $I(P)$ -module which we may consider as a left $I(P^{\text{op}})$ -module. Explicitly

$$\text{Hom}_{I(P)}(M, I(P))_x = \text{Hom}_{I(P)}(M, KP(x, -))$$

where the latter is the K -vector space of $I(P)$ -linear homomorphisms from M to the $I(P)$ -module $KP(x, -)$ with $KP(x, -)_y = KP(x, y) = K$ if $x \leq y$ and with $KP(x, -)_y = 0$ otherwise.

For $P = [0, \mathbf{t}]$ we shall denote the incidence algebra $I(P)$ by $\Lambda_{\mathbf{t}}$. There is an order preserving bijection $\tau: [0, \mathbf{t}] \rightarrow [0, \mathbf{t}]^{\text{op}}$ with $\tau(\mathbf{a}) = \mathbf{t} - \mathbf{a}$ and an induced homomorphism $\tau: \Lambda_{\mathbf{t}} \rightarrow \Lambda_{\mathbf{t}}^{\text{op}}$. Given a projective $\Lambda_{\mathbf{t}}$ -module M we shall be interested in the $\Lambda_{\mathbf{t}}$ -module $\tau^* \text{Hom}_{\Lambda_{\mathbf{t}}}(M, \Lambda_{\mathbf{t}})$.

Proposition 3.7. *Let $\mathbf{t} \in \mathbb{N}^n$. If M is a projective positively \mathbf{t} -determined S -module, then considering M and $D_{\mathbf{t}}(M)$ as $\Lambda_{\mathbf{t}}$ -modules there is a quasi-isomorphism $D_{\mathbf{t}}(M) \rightarrow T^n \tau^* \text{Hom}_{\Lambda_{\mathbf{t}}}(M, \Lambda_{\mathbf{t}})$. Moreover this quasi-isomorphism is natural in M .*

Proof. Since every indecomposable projective positively \mathbf{t} -determined S -module is isomorphic to $x^{\mathbf{a}}S$ for some $0 \leq \mathbf{a} \leq \mathbf{t}$ it suffices to consider the case $n = 1$. If M is the positively \mathbf{t} -determined S -module $M = x^a S = S(-a)$ corresponding to the $\Lambda_{\mathbf{t}}$ -module $KP(a, -)$, then $\text{Hom}_{\Lambda_{\mathbf{t}}}(M, \Lambda_{\mathbf{t}}) \cong KP(-, a)$, and thus $\tau^* \text{Hom}_{\Lambda_{\mathbf{t}}}(M, \Lambda_{\mathbf{t}})$ corresponds to the S -module $S(a - t)$. On the other hand, as we saw in Example 3.3, $D_t^1(M)$ is isomorphic to the chain complex

$$K_t\{0, t\} \rightarrow K_t\{0, t - a - 1\},$$

whose homology is $S(a - t)$ concentrated in homological degree one. \square

Lemma 3.8. *Let $\mathbf{t} \in \mathbb{N}^n$. If M is a positively \mathbf{t} -determined S -module, then considering M and $A_{\mathbf{t}}(M)$ as $\Lambda_{\mathbf{t}}$ -modules there is an isomorphism of the form $\tau^* \text{Hom}_K(M, K) \cong A_{\mathbf{t}}(M)$. Moreover this isomorphism is natural in M .*

Proof. Since $\text{Hom}_K(M, K)_{\mathbf{a}} = \text{Hom}_K(M_{\mathbf{a}}, K)$ we have

$$(\tau^* \text{Hom}_K(M, K))_{\mathbf{a}} = \text{Hom}_K(M_{\mathbf{t}-\mathbf{a}}, K) = A_{\mathbf{t}}(M)_{\mathbf{a}}$$

We leave it to the reader to check that this is an isomorphism of $\Lambda_{\mathbf{t}}$ -modules. \square

Combining the above two results we obtain the following description of the Nakayama functor.

Proposition 3.9. *If $F \rightarrow M$ is a projective resolution of a positively \mathbf{t} -determined S -module M , then considering F , $\mathcal{N}_{\mathbf{t}}^1(F)$ and $\mathcal{N}_{\mathbf{t}}^1(M)$ as $\Lambda_{\mathbf{t}}$ -modules there are quasi-isomorphisms of the form*

$$\tau^* \text{Hom}_K(\tau^* \text{Hom}_P(F, \Lambda_{\mathbf{t}}), K) \xrightarrow{\sim} T^n \mathcal{N}_{\mathbf{t}}^1(F) \xrightarrow{\sim} T^n \mathcal{N}_{\mathbf{t}}^1(M).$$

Proof. The first quasi-isomorphism follows directly from Corollary 3.5, Proposition 3.7, and Lemma 3.8. The second quasi-isomorphism is a consequence of Lemma 2.4 on exactness of the functors $M \mapsto p_{t_j}^* M$ and the fact that the chain complexes $P_{\mathbf{t}+\mathbf{1}}$ are projective over S . \square

3.6. The Auslander-Reiten translate. Given a self-injective Artin algebra Λ over K , the endofunctor

$$M \mapsto \text{Hom}_K(\text{Hom}_\Lambda(M, \Lambda), K)$$

of the category of left Λ modules, is termed the Nakayama functor by Auslander, Reiten and Smalø [1, p. 126], and they denote it by \mathcal{N} . More generally, if Λ is not self-injective we choose an injective bi-module resolution $\Lambda \rightarrow J$ of Λ and term the endofunctor

$$M \mapsto \text{Hom}_K(\text{Hom}_\Lambda(M, J), K)$$

the Nakayama functor.

In the situation $\Lambda = \Lambda_{\mathbf{t}}$ we have an isomorphism

$$\text{Hom}_K(\text{Hom}_{\Lambda_{\mathbf{t}}}(M, J), K) \cong \tau^* \text{Hom}_K(\tau^* \text{Hom}_{\Lambda_{\mathbf{t}}}(M, J), K).$$

Thus Proposition 3.9 implies that the Nakayama functor in the context of positively \mathbf{t} -determined S -modules corresponds to the Nakayama functor for the incidence algebra $\Lambda_{\mathbf{t}}$. On the bounded derived category of $\Lambda_{\mathbf{t}}$ -modules the Nakayama functor represents the Auslander–Reiten translate [6, p.37], since $\Lambda_{\mathbf{t}}$ has finite global dimension.

4. NAKAYAMA COHOMOLOGY OF MONOMIAL IDEALS

In this section we present the main results of this paper. Firstly, we calculate the multigraded cohomology and Betti spaces of iterations of the Nakayama functor applied to monomial quotient rings. The results of these calculations are expressed in terms of cohomology of certain simplicial complexes. Secondly, we describe the S -module structure of the cohomology modules of the Nakayama functor applied to monomial quotient rings. In order to specify the simplicial complexes occurring in our calculations we need some notation.

4.1. Simplicial complexes from monomial ideals. Let I be a positively \mathbf{t} -determined monomial ideal in S , and let $\mathbf{b} \leq \mathbf{t}$ and $\mathbf{a} \leq \mathbf{b} + \mathbf{1}$ in \mathbb{N}^n . The simplicial complex $\Delta_{\mathbf{a}}^{\mathbf{b}}(S/I; \mathbf{t})$ on the vertex set $\{1, \dots, n\}$ is defined to consist of the subsets F of the set $\{1, \dots, n\}$ with the property that the degree

$$\mathbf{x} = \sum_{i \notin F} a_i \varepsilon_i + \sum_{i \in F} (b_i + 1) \varepsilon_i$$

satisfies both $\mathbf{x} \leq \mathbf{t}$ and $(S/I)_{\mathbf{x}} = K$.

Example 4.1. Let I be a square free monomial ideal in S and let \mathbf{a} be the indicator vector of a subset A of $\{1, \dots, n\}$. The simplicial complex $\Delta = \Delta_0^0(S/I; \mathbf{1})$ is the simplicial complex with Stanley-Reisner ring equal to S/I . The simplicial complex $\Delta_{\mathbf{a}}^{\mathbf{a}}(S/I; \mathbf{1})$ is the link $\text{lk}_{\Delta}(A)$ of A in Δ , that is, the simplicial complex consisting of the faces F in Δ satisfying that $F \cap A = \emptyset$ and $F \cup A \in \Delta$. Finally $\Delta_0^{1-\mathbf{a}}(S/I; \mathbf{1})$ is the restriction $\Delta|_A$ consisting of the faces F in Δ contained in A .

If \mathbf{t} and the positively \mathbf{t} -determined ideal I are given explicitly we shall also denote the simplicial complex $\Delta_{\mathbf{a}}^{\mathbf{b}}(S/I; \mathbf{t})$ simply as $\Delta_{\mathbf{a}}^{\mathbf{b}}$. The following lemma records a combinatorial relation between the simplicial complexes $\Delta_{\mathbf{a}}^{\mathbf{b}}$ which is crucial for our calculations.

Lemma 4.2. *Let I be a positively \mathbf{t} -determined monomial ideal in S , and $\mathbf{b} \leq \mathbf{t}$ and $\mathbf{a} \leq \mathbf{b} + \mathbf{1}$ in \mathbb{N}^n . Given $\alpha, \beta \in \mathbb{N}$ with $a_j + \alpha \leq b_j + 1$. Then*

$$\Delta_{\mathbf{a}}^{\mathbf{b}} = \Delta_{\mathbf{a}}^{\mathbf{b}+\beta\epsilon_j} \cup \Delta_{\mathbf{a}+\alpha\epsilon_j}^{\mathbf{b}}$$

and

$$\Delta_{\mathbf{a}+\alpha\epsilon_j}^{\mathbf{b}+\beta\epsilon_j} = \Delta_{\mathbf{a}}^{\mathbf{b}+\beta\epsilon_j} \cap \Delta_{\mathbf{a}+\alpha\epsilon_j}^{\mathbf{b}}.$$

Proof. This is seen by splitting into the cases whether a subset F of $\{1, 2, \dots, n\}$ contains j or not. \square

Given a simplicial complex Δ we denote by $\tilde{H}^*(\Delta)$ its reduced cohomology with coefficients in K . The above lemma implies that there is a Mayer–Vietoris exact sequence of reduced cohomology groups.

Corollary 4.3. *Let I be a positively \mathbf{t} -determined monomial ideal in S , and let $\mathbf{b} \leq \mathbf{t}$ and $\mathbf{a} \leq \mathbf{b} + \mathbf{1}$ in \mathbb{N}^n . Given $\alpha, \beta \in \mathbb{N}$ with $a_j + \alpha \leq b_j + 1$. Then there is a Mayer–Vietoris exact sequence of the form*

$$\dots \rightarrow \tilde{H}^i(\Delta_{\mathbf{a}}^{\mathbf{b}}) \rightarrow \tilde{H}^i(\Delta_{\mathbf{a}}^{\mathbf{b}+\beta\epsilon_j}) \oplus \tilde{H}^i(\Delta_{\mathbf{a}+\alpha\epsilon_j}^{\mathbf{b}}) \rightarrow \tilde{H}^i(\Delta_{\mathbf{a}+\alpha\epsilon_j}^{\mathbf{b}+\beta\epsilon_j}) \xrightarrow{\delta^i} \tilde{H}^{i+1}(\Delta_{\mathbf{a}}^{\mathbf{b}}) \rightarrow \dots$$

4.2. Nakayama cohomology. We are now going to compute the cohomology of $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ for every positively \mathbf{t} -determined ideal I in S . In Example 2.12 we have treated the case $n = 1$. Note from this that for every interval $[a, b]$ contained in $[0, t]$ there are easily computable γ, x and y such that $\mathcal{N}_t^{\mathbf{k}}(K_t\{a, b\})$ is quasi-isomorphic to $T^{-\gamma}K_t\{x, y\}$. For $n > 1$ we have the isomorphism

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\}) \cong \bigotimes_{j=1}^n \mathcal{N}_{t_j}^{k_j}(K_{t_j}\{a_j, b_j\})$$

where the outer tensor product is over K . Therefore we can easily compute γ, \mathbf{x} and \mathbf{y} such that $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(K_{\mathbf{t}}\{\mathbf{a}, \mathbf{b}\})$ is quasi-isomorphic to $T^{-\gamma}(K_{\mathbf{t}}\{\mathbf{x}, \mathbf{y}\})$.

Theorem 4.4. *Let $\mathbf{t}, \mathbf{k} \in \mathbb{N}^n$ and let $\mathbf{r} \in [0, \mathbf{t}]$. Moreover let \mathbf{u}, \mathbf{v} in \mathbb{N}^n and $\gamma \in \mathbb{Z}$ be such that the only nonzero cohomology group*

$$H^{\gamma+n}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}K_{\mathbf{t}}\{\mathbf{t} - \mathbf{r}, \mathbf{t}\} \cong K_{\mathbf{t}}\{\mathbf{t} - \mathbf{v}, \mathbf{t} - \mathbf{u}\}.$$

There is an isomorphism

$$H^i\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I; \mathbf{t}))$$

for every positively \mathbf{t} -determined ideal I in S .

Remark 4.5. Note that $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}K_{\mathbf{t}}\{\mathbf{t} - \mathbf{r}, \mathbf{t}\}$ is quasi-isomorphic to $T^{-n}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}K_{\mathbf{t}}\{\mathbf{0}, \mathbf{t} - \mathbf{r}\}$ and to determine \mathbf{u}, \mathbf{v} and γ we shall use the latter. We have stated the theorem as above to emphasize the analogy with Theorem 4.6 below.

Let I be a positively \mathbf{t} -determined ideal in S . The following theorem describes the Betti-spaces $B^p(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I))$, which may be computed as $H^p(F \otimes_S \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I))$ for F a projective \mathbb{N}^n -graded resolution of K over S .

Theorem 4.6. *Let $\mathbf{t}, \mathbf{k} \in \mathbb{N}^n$ and let $\mathbf{r} \in [0, \mathbf{t}]$. Moreover let \mathbf{a}, \mathbf{b} in \mathbb{N}^n and $\gamma \in \mathbb{Z}$ be such that the only nonzero cohomology group*

$$H^{\gamma+n}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}K_{\mathbf{t}}\{\mathbf{t} - \mathbf{r}, \mathbf{t} - \mathbf{r}\} \cong K_{\mathbf{t}}\{\mathbf{t} - \mathbf{b}, \mathbf{t} - \mathbf{a}\}.$$

There is an isomorphism

$$B^i(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I))_{\mathbf{r}} \cong \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{a}}^{\mathbf{b}}(S/I; \mathbf{t}))$$

for every positively \mathbf{t} -determined ideal I in S .

It is easy to give an explicit description of \mathbf{u} , \mathbf{v} and γ in Theorem 4.4. In order to do this we first note that in view of Proposition 2.16 it suffices to consider the case where $\mathbf{k} \leq \mathbf{t} + \mathbf{1}$.

Definition 4.7. Given $r, k, t \in \mathbb{N}$ with $k \leq t + 1$ we define $\gamma_t^k(r)$, $u_t^k(r)$ and $v_t^k(r)$ by the formulas

$$\gamma_t^k(r) = \begin{cases} 0 & \text{if } k \leq r \\ 1 & \text{if } r + 1 \leq k, \end{cases}$$

$$u_t^k(r) = \begin{cases} r - k & \text{if } k \leq r \\ t - k + 1 & \text{if } r + 1 \leq k \end{cases}$$

and

$$v_t^k(r) = \begin{cases} t - k & \text{if } k \leq r \\ t - k + r + 1 & \text{if } r + 1 \leq k. \end{cases}$$

Let $\gamma_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r}) = \sum_{j=1}^n \gamma_{t_j}^{k_j}(r_j)$, let $\mathbf{u}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r}) = (u_{t_1}^{k_1}(r_1), \dots, u_{t_n}^{k_n}(r_n))$, and let $\mathbf{v}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r}) = (v_{t_1}^{k_1}(r_1), \dots, v_{t_n}^{k_n}(r_n))$. In Section 6.2 we prove the following.

Proposition 4.8. The elements $\mathbf{u} = \mathbf{u}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ and $\mathbf{v} = \mathbf{v}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ in \mathbb{N}^n and the integer $\gamma = \gamma_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ satisfy the first equation in Theorem 4.4.

In the particular case $\mathbf{k} = \mathbf{1}$ we recover Takayama's calculation of local cohomology [11].

Corollary 4.9 (Takayama). Let $\mathbf{t} \in \mathbb{N}^n$ and $\mathbf{r} \in [0, \mathbf{t}]$. With the notation $\mathbf{u} = \mathbf{u}_{\mathbf{t}}^{\mathbf{1}}(\mathbf{r})$, $\mathbf{v} = \mathbf{v}_{\mathbf{t}}^{\mathbf{1}}(\mathbf{r})$ and $\gamma = \gamma_{\mathbf{t}}^{\mathbf{1}}(\mathbf{r})$ there are isomorphisms of the form

$$H_{\mathbf{m}}^i(S/I)_{\mathbf{r}-\mathbf{1}} \cong H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{1}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I; \mathbf{t})).$$

for every positively \mathbf{t} -determined ideal I in S .

Proof. The first isomorphism is a direct consequence of Proposition 2.7. The second isomorphism is just a special case of Proposition 4.8 and Theorem 4.4. \square

The above corollary on the other hand is an extension of Hochster's calculation [7] of local cohomology of square-free monomial ideals, or in other words, of $\mathbf{1}$ -determined ideals in S .

We now proceed to describe the S -module structure of the Nakayama cohomology groups $H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ of a positively \mathbf{t} -determined ideal I in S . Since this S -module is positively \mathbf{t} -determined it suffices to describe the multiplication maps

$$x_j: H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}-\varepsilon_j} \rightarrow H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}}$$

whenever $\varepsilon_j \leq \mathbf{r} \leq \mathbf{t}$.

Theorem 4.10. Let I be a positively \mathbf{t} -determined ideal in S and let $\mathbf{r} \in \mathbb{N}^n$ with $\varepsilon_j \leq \mathbf{r} \leq \mathbf{t}$. Using $\Delta_{\mathbf{a}}^{\mathbf{b}}$ as short hand notation for $\Delta_{\mathbf{a}}^{\mathbf{b}}(S/I; \mathbf{t})$ the multiplication map

$$x_j: H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}-\varepsilon_j} \rightarrow H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}}$$

can be described as follows:

- (1) If $k_j \neq r_j$, then $\Delta_{\mathbf{u}_t^{\mathbf{k}(\mathbf{r})}}^{\mathbf{v}_t^{\mathbf{k}(\mathbf{r})}}$ is a subcomplex of $\Delta_{\mathbf{u}_t^{\mathbf{k}(\mathbf{r}-\varepsilon_j)}}^{\mathbf{v}_t^{\mathbf{k}(\mathbf{r}-\varepsilon_j)}}$ and multiplication by x_j corresponds to the map of cohomology groups induced by this inclusion.
- (2) If $k_j = r_j$, then multiplication by x_j corresponds to $(-1)^{\sum_{i=1}^{j-1} \gamma_i(r_i)}$ times the Mayer-Vietoris connecting homomorphism $\delta^{i-\gamma-2}$ of Corollary 4.3 with $\mathbf{a} = \mathbf{u}_t^{\mathbf{k}(\mathbf{r})}$, $\mathbf{b} = \mathbf{v}_t^{\mathbf{k}(\mathbf{r})}$, $\gamma = \gamma_t^{\mathbf{k}(\mathbf{r})}$, $\alpha = t_j - r_j + 1$ and $\beta = r_j$.

The case $\mathbf{k} = \mathbf{t} = \mathbf{1}$ of the above theorem is due to Gräbe [5] as we shall now explain. In this case I is $\mathbf{1}$ -determined, and thus S/I is the Stanley-Reisner ring of the simplicial complex $\Delta = \Delta(S/I)$. Note that in this case $\varepsilon_j \leq \mathbf{r} \leq \mathbf{t} = \mathbf{1}$ implies $r_j = 1 = k_j$, and thus the S -module structure of $H^i \mathcal{N}_1^1(S/I)$ is given by Mayer-Vietoris boundary maps. Further $\mathbf{v}_1^1(\mathbf{r}) = \mathbf{u}_1^1(\mathbf{r}) = \mathbf{1} - \mathbf{r}$ (check!) for every $\mathbf{r} \in [0, 1]$. If we let \overline{R} denote the subset $\overline{R} = \{j : r_j = 0\}$ of $\{1, \dots, n\}$ then we saw in Example 4.1 that $\Delta_{\mathbf{u}_1^1(\mathbf{r})}^{\mathbf{v}_1^1(\mathbf{r})}(S/I; \mathbf{1})$, which is $\Delta_{\mathbf{1}-\mathbf{r}}^{\mathbf{1}-\mathbf{r}}(S/I; \mathbf{1})$, is the link $\text{lk}_\Delta(\overline{R})$ of \overline{R} in Δ . Gräbe considers the contrastar $\text{cost}_\Delta(\overline{R})$ of \overline{R} in Δ given as

$$\text{cost}_\Delta(\overline{R}) := \{G \in \Delta : \overline{R} \not\subseteq G\},$$

and he notes that there is an isomorphism

$$\begin{aligned} T^{|\overline{R}|} \tilde{C}_*(\text{lk}_\Delta(\overline{R})) &\cong \tilde{C}_*(\Delta) / \tilde{C}_*(\text{cost}_\Delta(\overline{R})), \\ G &\mapsto (-1)^{a(\overline{R}, G)} (\overline{R} \cup G) + \tilde{C}_*(\text{cost}_\Delta(\overline{R})), \end{aligned}$$

where $a(\overline{R}, G)$ is the cardinality of $\{(f, g) \in \overline{R} \times G : f < g\}$. On the level of cohomology groups we obtain a commutative diagram of the form

$$\begin{array}{ccc} \tilde{H}^{i-|\overline{R}|-1}(\text{lk}_\Delta(\overline{R} \cup \{j\})) & \xrightarrow{\cong} & \tilde{H}^i(\Delta, \text{cost}_\Delta(\overline{R} \cup \{j\})) \\ (-1)^{a(\overline{R}, \{j\})} \delta \downarrow & & \downarrow \\ \tilde{H}^{i-|\overline{R}|}(\text{lk}_\Delta(\overline{R})) & \xrightarrow{\cong} & \tilde{H}^i(\Delta, \text{cost}_\Delta(\overline{R})). \end{array}$$

Here δ is the Mayer-Vietoris connecting homomorphism of Corollary 4.3

$$\tilde{H}^{i-|\overline{R}|-1}(\Delta_{\mathbf{1}-\mathbf{r}+\varepsilon_j}^{\mathbf{1}-\mathbf{r}+\varepsilon_j}(S/I; \mathbf{1})) \rightarrow \tilde{H}^{i-|\overline{R}|}(\Delta_{\mathbf{1}-\mathbf{r}}^{\mathbf{1}-\mathbf{r}}(S/I; \mathbf{1}))$$

(in the situation $\mathbf{a} = \mathbf{b} = \mathbf{1} - \mathbf{r}$ and $\alpha = \beta = 1$). The right hand vertical homomorphism is induced from the inclusion $\text{cost}_\Delta(\overline{R}) \subseteq \text{cost}_\Delta(\overline{R} \cup \{j\})$. Since $H^i \mathcal{N}_1^1(S/I)$ is isomorphic to $p_1^*(H_m^i(S/I)(-\mathbf{1}))$ we recover Gräbe's result [5] in the following formulation.

Theorem 4.11 (Gräbe, 1984). *Let $I \subseteq S$ be a square-free monomial ideal and let $\Delta = \Delta(S/I)$. For every $\mathbf{r} \in \{0, 1\}^n$ there is, with the notation $\overline{R} = \{j : r_j = 0\}$, an isomorphism*

$$H_m^i(S/I)_{\mathbf{r}-\mathbf{1}} \cong \tilde{H}^{i-1}(\Delta, \text{cost}_\Delta(\overline{R})).$$

If $\varepsilon_j \leq \mathbf{r} \leq \mathbf{1}$, then under the above isomorphism the multiplication

$$x_j : H_m^i(S/I)_{\mathbf{r}-\varepsilon_j-\mathbf{1}} \rightarrow H_m^i(S/I)_{\mathbf{r}-\mathbf{1}}$$

corresponds to the homomorphism of cohomology groups induced by the inclusion $\text{cost}_\Delta(\overline{R}) \subseteq \text{cost}_\Delta(\overline{R} \cup \{j\})$.

5. VANISHING OF NAKAYAMA COHOMOLOGY

In this section we give conditions for the vanishing or non-vanishing of the cohomology groups of the Nakayama functor $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$. Towards the end we discuss the simple case when the polynomial ring S is in two variables, and give the conditions for when the Nakayama functor has exactly one non-vanishing cohomology group. The ideal I is assumed to be positively \mathbf{t} -determined.

5.1. General results.

Lemma 5.1. *The Nakayama cohomology groups $H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanish for $i \geq 2n$ and for $i < 0$.*

Proof. Theorem 4.4 gives the isomorphism

$$H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I; \mathbf{t})).$$

where $\gamma = \gamma_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$, $\mathbf{u} = \mathbf{u}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ and $\mathbf{v} = \mathbf{v}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ are specified in Proposition 4.8. In particular note that $0 \leq \gamma \leq n$. Since $\Delta_{\mathbf{u}}^{\mathbf{v}}$ is a simplicial complex on n vertices, the reduced homology group $\tilde{H}^j(\Delta_{\mathbf{u}}^{\mathbf{v}})$ can be nonvanishing only in the range $-1 \leq j \leq n-2$. Hence the right side above can only be nonvanishing in the range $-1 \leq i - \gamma - 1 \leq n - 2$. But since $0 \leq \gamma \leq n$ this gives $0 \leq i \leq 2n - 1$. \square

Definition 5.2. Let I be a positively \mathbf{t} -determined ideal and $\mathbf{y} \in [0, \mathbf{t}]$.

- S/I has a *peak* at \mathbf{y} if $(S/I)_{\mathbf{y}} \neq 0$ and $(S/I)_{\mathbf{x}} = 0$ for all $\mathbf{y} < \mathbf{x} \leq \mathbf{t}$.
- S/I has an *indent* at \mathbf{y} if $(S/I)_{\mathbf{y}} = 0$ and $(S/I)_{\mathbf{x}} \neq 0$ for all $0 \leq \mathbf{x} < \mathbf{y}$.
- A peak \mathbf{y} is said to be of relative dimension m below \mathbf{x} if $\mathbf{y} \leq \mathbf{x}$ and the number of non-zero coordinates of $\mathbf{x} - \mathbf{y}$ is m . An indent \mathbf{y} is said to be of relative dimension m above \mathbf{x} if $\mathbf{y} \geq \mathbf{x}$ and the number of nonzero coordinates of $\mathbf{y} - \mathbf{x}$ is m .

The following gives sufficient criteria for the nonvanishing of cohomology groups. They are however by no means necessary. The vanishing of cohomology groups is of course more interesting and the following lemma is therefore a step towards Proposition 5.5.

Lemma 5.3. *Let I be a positively \mathbf{t} -determined ideal in S and $1 \leq \mathbf{k} \leq \mathbf{t} + 1$.*

- If S/I has a peak of relative dimension m below $\mathbf{t} + 1 - \mathbf{k}$, then $H^{n-m} \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ is non-zero.*
- If S/I has an indent of relative dimension m above $\mathbf{t} + 1 - \mathbf{k}$, then $H^{n-1+m} \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ is non-zero.*

Proof. a. Let \mathbf{u} be the peak. Define $\mathbf{v} \in [0, \mathbf{t}]$ by letting

$$v_j = \begin{cases} t_j - k_j & \text{if } t_j - k_j + 1 > u_j \\ t_j & \text{if } t_j - k_j + 1 = u_j. \end{cases}$$

Moreover define $\mathbf{r} \in [0, \mathbf{t}]$ by letting

$$r_j = \begin{cases} u_j + k_j & \text{if } t_j - k_j + 1 > u_j \\ k_j - 1 & \text{if } t_j - k_j + 1 = u_j. \end{cases}$$

Then $\gamma_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r}) = n - m$, $\mathbf{u} = \mathbf{u}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ and $\mathbf{v} = \mathbf{v}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$. Thus Theorem 4.4 and Proposition 4.8 imply that

$$H^{n-m} \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)).$$

Since \mathbf{u} is a peak of relative dimension m below $\mathbf{t} + \mathbf{1} - \mathbf{k}$, the simplicial complex $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ contains only the empty subset of $\{1, \dots, n\}$, and therefore the right side above is isomorphic to K , and so $H^{n-m}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ is nonvanishing.

b. Let \mathbf{y} be the indent and set $\mathbf{v} = \mathbf{y} - \mathbf{1}$. Define $\mathbf{u} \in [0, \mathbf{t}]$ by letting

$$u_j = \begin{cases} 0 & \text{if } v_j = t_j - k_j \\ t_j - k_j + 1 & \text{if } v_j > t_j - k_j. \end{cases}$$

Moreover define $\mathbf{r} \in [0, \mathbf{t}]$ by letting

$$r_j = \begin{cases} k_j & \text{if } v_j = t_j - k_j \\ v_j - t_j + k_j - 1 & \text{if } v_j > t_j - k_j. \end{cases}$$

Then $\gamma_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r}) = m$, $\mathbf{u} = \mathbf{u}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ and $\mathbf{v} = \mathbf{v}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$. Thus Theorem 4.4 and Proposition 4.8 imply that

$$H^{n-1+m}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{n-2}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)).$$

Since \mathbf{v} is an indent of relative dimension m above $\mathbf{t} + \mathbf{1} - \mathbf{k}$, the simplicial complex $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is the simplicial $(n-2)$ -sphere consisting of all proper subsets of $\{1, \dots, n\}$. Then the right side above is isomorphic to K and so $H^{n-1+m}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ is nonvanishing. \square

Corollary 5.4. *If I is a positively \mathbf{t} -determined ideal in S and S/I has a peak of relative dimension m below \mathbf{t} , then $H_{\mathbf{m}}^{n-m}(S/I) \neq 0$.*

In the extremal cases $m = n$ we can characterize exactly when the Nakayama cohomology vanishes.

Proposition 5.5. *Let I be a positively \mathbf{t} -determined ideal and $\mathbf{1} \leq \mathbf{k} \leq \mathbf{t} + \mathbf{1}$.*

a. *The zero'th Nakayama cohomology group $H^0\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes if and only if no peak of S/I is contained in $[0, \mathbf{t} - \mathbf{k}]$.*

b. *The $(2n-1)$ -th Nakayama cohomology group $H^{2n-1}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes if and only if no indent of S/I is greater or equal to $\mathbf{t} + 2 \cdot \mathbf{1} - \mathbf{k}$, save an exception when $n = 1$ and $I = 0$.*

Proof. a. Suppose the zero'th Nakayama cohomology group does not vanish in degree \mathbf{r} . Since Theorem 4.4 gives

$$H^0\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)),$$

we must have $\gamma = 0$ and the simplicial complex on the right side must be $\{\emptyset\}$. Moreover Proposition 4.8 implies that $\mathbf{k} \leq \mathbf{r}$, that $\mathbf{u} = \mathbf{r} - \mathbf{k}$ and that $\mathbf{v} = \mathbf{t} - \mathbf{k}$. Now the fact that $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is $\{\emptyset\}$ implies that S/I has a peak in the interval $[\mathbf{u}, \mathbf{v}]$ which is of relative dimension n below $\mathbf{t} + \mathbf{1} - \mathbf{k}$.

Conversely, if S/I has a peak of relative dimension n below $\mathbf{t} + \mathbf{1} - \mathbf{k}$, then Proposition 5.3a. implies the non-vanishing of the zeroth Nakayama cohomology group.

b. Suppose the $2n-1$ 'th Nakayama cohomology group is non-vanishing in degree \mathbf{r} . Since Theorem 4.4 gives

$$H^{2n-1}\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong \tilde{H}^{2n-1-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)),$$

and since $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is a simplicial complex on the vertex set $\{1, \dots, n\}$ we have $2n-1-\gamma-1 \leq n-2$, that is, $n \leq \gamma$. By Proposition 4.8, γ can be at most

n and so $\gamma = n$. By the same Proposition 4.8 we get $\mathbf{u} = \mathbf{t} - \mathbf{k} + \mathbf{1}$ and that $\mathbf{v} = \mathbf{t} - \mathbf{k} + \mathbf{r} + \mathbf{1}$. Therefore $\tilde{H}^{n-2}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I))$ is nonvanishing, and consequently $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is the simplicial sphere consisting of all proper subsets of $\{1, \dots, n\}$. If $n > 1$ we conclude that $\mathbf{v} + \mathbf{1} \leq \mathbf{t}$. If $n = 1$ we can also conclude that $v + 1 \leq t$ provided $r \leq k - 2$. Assume then that $\mathbf{v} + \mathbf{1} \leq \mathbf{t}$. Consider the elements \mathbf{x} in $[\mathbf{u}, \mathbf{v} + \mathbf{1}]$ such that $(S/I)_{\mathbf{x}}$ is nonzero. Note that i) $(S/I)_{\mathbf{v} + \mathbf{1}}$ is zero, and ii) if $x_i = u_i$ for some i , then $(S/I)_{\mathbf{x}}$ is nonzero. Therefore a minimal element in $[\mathbf{u}, \mathbf{v} + \mathbf{1}]$ such that $(S/I)_{\mathbf{x}}$ is zero must exist and be in the interval $[\mathbf{u} + \mathbf{1}, \mathbf{v} + \mathbf{1}]$. As such it is an indent of relative dimension n above \mathbf{u} .

The only case left is when $n = 1$ and we must have $r \geq k - 1$. Then we see that $(S/I)_t$ must be nonzero, and so $I = 0$. In this case we see that the above cohomology group is always nonzero in degree $r = k - 1$.

Conversely, if S/I has indent, Proposition 5.3b. implies the non-vanishing of the $2n - 1$ 'th Nakayama cohomology group. \square

5.2. The case of two variables. We now assume that $S = K[x, y]$, the polynomial ring in two variables. In this case there can only be four possible nonvanishing cohomology modules of $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ and these are in degrees 0, 1, 2, and 3. We assume in this subsection that $\mathbf{1} \leq \mathbf{k} \leq \mathbf{t} + \mathbf{1}$.

Lemma 5.6. *Let I be a positively \mathbf{t} -determined ideal.*

a. *The zero'th cohomology group $H^0 \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes if and only if the support of S/I in $[\mathbf{0}, \mathbf{t} + \mathbf{1} - \mathbf{k}]$ is the complement of a (possibly empty) interval.*

b. *Its third cohomology group $H^3 \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes if and only if the support of S/I in $[\mathbf{t} + \mathbf{1} - \mathbf{k}, \mathbf{t}]$ is a (possibly empty) interval.*

Proof. This is just a translation of Proposition 5.5 when we take into consideration that the intervals are in the plane. \square

Lemma 5.7. *The second cohomology group $H^2 \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes if and only if S/I is zero in degree $\mathbf{t} + \mathbf{1} - \mathbf{k}$.*

Proof. Suppose $(S/I)_{\mathbf{t} + \mathbf{1} - \mathbf{k}}$ is nonzero. Letting $\mathbf{r} = \mathbf{k} - \mathbf{1}$, Proposition 4.8 gives $\gamma = 2$, $\mathbf{u} = \mathbf{t} - \mathbf{k} + \mathbf{1}$, and $\mathbf{v} = \mathbf{t}$. Then $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is $\{\emptyset\}$ and so its \tilde{H}^{-1} cohomology is K . By Theorem 4.4, the second cohomology group in the lemma statement does not vanish.

Conversely, suppose $H^2 \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ does not vanish in degree \mathbf{r} . By Theorem 4.4, $\tilde{H}^{1-\gamma}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I))$ is nonvanishing. This is the reduced cohomology group of a simplicial complex on two vertices, and so either i) $1 - \gamma = -1$, giving $\gamma = 2$, or ii) $1 - \gamma = 0$ giving $\gamma = 1$.

i) Then \mathbf{u} is $\mathbf{t} + \mathbf{1} - \mathbf{k}$ and $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is $\{\emptyset\}$, implying S/I nonzero in degree \mathbf{u} .

ii) Then \mathbf{u} is $(r_1 - k_1, t_2 + 1 - k_2)$ (or the analog with coordinates switched), and \mathbf{v} is $(t_1 - k_1, t_2 + 1 - k_2 + r_2)$. Since $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ in this case must consist of two points, the definition of this simplicial complex implies that S/I is nonzero in degree $\mathbf{t} + \mathbf{1} - \mathbf{k}$. \square

Lemma 5.8. *The first cohomology group $H^1 \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes if and only if either i) S/I is nonzero in degree $\mathbf{t} + \mathbf{1} - \mathbf{k}$ or ii) the degrees for which S/I is nonzero is contained in $[\mathbf{0}, \mathbf{t} - \mathbf{k}]$.*

Proof. Suppose this first cohomology group is nonvanishing in degree \mathbf{r} . By Theorem 4.4, $\tilde{H}^{-\gamma}(\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I))$ is nonvanishing. This is the reduced cohomology group of a simplicial complex in two variables, and so either i) $-\gamma = -1$, or ii) $-\gamma = 0$.

i) In this case \mathbf{u} is $(r_1 - k_1, t_2 + 1 - k_2)$ (or the analog with the coordinates switched), \mathbf{v} is $(t_1 - k_1, t_2 + 1 + r_2 - k_2)$ and $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ is $\{\emptyset\}$. By the definition of this simplicial complex, S/I is nonzero in degree \mathbf{u} and zero in degree $\mathbf{t} + \mathbf{1} - \mathbf{k}$.

ii) In this case \mathbf{u} is $(r_1 - k_1, r_2 - k_2)$ and \mathbf{v} is $(t_1 - k_1, t_2 - k_2)$ and $\Delta_{\mathbf{u}}^{\mathbf{v}}(S/I)$ consists of two points. Hence by its definition, S/I is nonzero in degree, say $(t_1 - k_1 + 1, r_2 - k_2)$, and zero in degree $\mathbf{t} + \mathbf{1} - \mathbf{k}$.

Conversely, suppose S/I is zero in degree $\mathbf{t} + \mathbf{1} - \mathbf{k}$ and the support of S/I is not contained in $[\mathbf{0}, \mathbf{t} - \mathbf{k}]$. If $(t_1 + 1 - k_1, 0)$ and $(0, t_2 + 1 - k_2)$ are both in the support of S/I we can realize case ii) above for $\mathbf{r} = \mathbf{k}$. If, say only the second is in the support, we can realize case i) above with $\mathbf{r} = \mathbf{k} + \epsilon_2$. \square

We can now summarize these results as follows.

Proposition 5.9. *Suppose S is a polynomial ring in two variables and $\mathbf{k} \geq \mathbf{1}$. Then $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ has at most one nonvanishing cohomology group if and only if either*

- a. *The support of S/I is contained in $[\mathbf{0}, \mathbf{t} - \mathbf{k}]$.*
- b. *The support of S/I in $[\mathbf{t} + \mathbf{1} - \mathbf{k}, \mathbf{t}]$ is a nonempty interval.*
- c. *The support of S/I in $[\mathbf{0}, \mathbf{t} + \mathbf{1} - \mathbf{k}]$, is the complement of a nonempty interval.*

Proof. Suppose $\mathbf{t} + \mathbf{1} - \mathbf{k}$ is in the support of S/I , so cases a. and c. do not apply. Then H^0 and H^1 of $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ vanishes, while H^3 vanishes if and only if we are in case b.

Suppose $\mathbf{t} + \mathbf{1} - \mathbf{k}$ is not in the support of S/I , so case b. does not apply. Then H^2 and H^3 of the Nakayama functor vanishes. That H^1 vanishes is equivalent to case a., and that H^0 vanishes is equivalent to case c. \square

6. THE NAKAYAMA FUNCTOR ON INTERVAL MODULES

In this section we consider the Nakayama functor applied to interval modules. We investigate quite explicitly and in detail the one variable case $n = 1$, Subsection 6.1. In particular we develop lemmata which will be pivotal in the final proofs. In Subsection 6.2 we infer immediate consequences to the general setting $n \geq 1$. Subsection 6.3 concerns the explicit description of a connecting homomorphism needed for the proof of Theorem 4.10.

6.1. One dimensional intervals. In order to keep track of naturality properties of the Nakayama functor we need to be quite explicit in our calculation of one-dimensional Nakayama cohomology.

Definition 6.1. Let $k, t \in \mathbb{N}$ with $k \leq t + 1$. Given $y \in [0, t]$ we define the chain complex $N_t^k(t - y)$ of t -determined S -modules concentrated in degrees 0 and -1 , where it is given by the formula

$$N_t^k(t - y) = \begin{cases} K_t\{k, t - y + k\} \rightarrow 0 & \text{if } k \leq y \\ K_t\{k, t\} \xrightarrow{i} K_t\{k - y - 1, t\} & \text{if } y + 1 \leq k. \end{cases}$$

Here i is the canonical inclusion introduced in Example 2.6. For $0 \leq y \leq y' \leq t$ there is a naturally defined chain homomorphism

$$N_t^k(y, y') : N_t^k(t - y) \rightarrow N_t^k(t - y')$$

as follows:

- (1) If $k \leq y$, then $N_t^k(y, y')$ is the canonical projection $p: K_t\{k, t - y + k\} \rightarrow K_t\{k, t - y' + k\}$ introduced in Example 2.6.
- (2) If $y < k \leq y'$, then $N_t^k(y, y')$ in degree zero is the canonical projection $p: K_t\{k, t\} \rightarrow K_t\{k, t - y' + k\}$.
- (3) If $y' < k$, then $N_t^k(y, y')$ is the identity in degree zero and the canonical inclusion $i: K_t\{k - y - 1, t\} \rightarrow K_t\{k - y' - 1, t\}$ in degree -1 .

The following lemma gives an explicit description of the Nakayama functor applied to S/I for a graded ideal I in $S = K[x]$.

Lemma 6.2. *For $0 \leq k \leq t + 1$ and $0 \leq y \leq t$ there is a quasi-isomorphism*

$$\mathcal{N}_t^k K_t\{0, t - y\} \xrightarrow{f_t^k(y)} N_t^k(t - y)$$

of chain-complexes of S -modules. Further, if $y \leq y'$, then

$$N_t^k(y, y') \circ f_t^k(y) = f_t^k(y') \circ \mathcal{N}_t^k(p),$$

where $p: K_t\{0, t - y\} \rightarrow K_t\{0, t - y'\}$ is the canonical projection.

Proof. We prove the result by induction on k . In Example 2.12 we saw that

$$\mathcal{N}_t^1 K_t\{a, b\} \cong \begin{cases} K_t\{a + 1, b + 1\} & \text{if } b < t \\ K_t\{a + 1, t\} \rightarrow K_t\{0, t\} & \text{if } b = t \end{cases}$$

In particular $\mathcal{N}_t^1 K_t\{0, t - y\} \cong N_t^1(t - y)$, so the lemma holds for $k = 1$. Next suppose that the lemma holds for $k - 1$. In order to see that it holds for k it suffices to provide a quasi-isomorphism $g(y): \mathcal{N}_t^1 N_t^{k-1}(t - y) \rightarrow N_t^k(t - y)$ such that

$$N_t^k(y, y') \circ g(y) = g(y') \circ \mathcal{N}_t^1 N_t^{k-1}(y, y')$$

when $y \leq y'$. In the case $k \leq y + 1$ this is very easy because the two chain complexes are isomorphic. Otherwise, if $k \geq y + 2$, then $\mathcal{N}_t^1 N_t^{k-1}(t - y)$ is the total complex of the bicomplex

$$\begin{array}{ccc} K_t\{k, t\} & \longrightarrow & K_t\{k - y - 1, t\} \\ \downarrow & & \downarrow \\ K_t\{0, t\} & \xrightarrow{=} & K_t\{0, t\}. \end{array}$$

Since $N_t^k(t - y)$ is equal to the top row of this bicomplex there is a quasi-isomorphism $g(y): \mathcal{N}_t^1 N_t^{k-1}(t - y) \rightarrow N_t^k(t - y)$ projecting onto the top row. \square

The above lemma enables us to verify Proposition 4.8 in the case $n = 1$.

Corollary 6.3. *For $0 \leq k \leq t + 1$ and $0 \leq y \leq t$ the cohomology of the chain complex $\mathcal{N}_t^k K_t\{0, t - y\}$ is isomorphic to $K_t\{t - v_t^k(y), t - u_t^k(y)\}$ concentrated in cohomological degree $\gamma_t^k(y)$.*

The following somewhat surprising result is the main input in our proof in Section 9 of Theorem 4.4, which computes the cohomology of the Nakayama functor.

Lemma 6.4. *For every u with $0 \leq u \leq t$ and every y with $0 \leq y \leq t$ there is a quasi-isomorphism*

$$N_t^k(t - y)_u \xrightarrow{g_t^k(y, u)} N_t^k(t - u)_y.$$

Moreover for every chain $d \in N_t^k(t - y)_u$ and all y, y' with $0 \leq y \leq y' \leq t$ we have

$$x^{y'-y} \cdot g_t^k(y, u)(d) = g_t^k(y', u)(N_t^k(y, y')(d))$$

Proof. We define $g_t^k(y, u)$ to be the identity on K whenever

$$N_t^k(t - y)_u = N_t^k(t - u)_y = K$$

and to be zero otherwise. When $k = t + 1$ the assertions of the lemma are obvious. When $k \leq t$ the assertions can be verified by splitting up into the six cases obtained from the three cases $k \leq y \leq y'$ or $y < k \leq y'$ or $y \leq y' < k$ times the two cases $k \leq u$ or $u < k$. \square

Corollary 6.5. *For all u and all y, y' with $0 \leq y \leq y' \leq t$ the following diagram commutes.*

$$\begin{array}{ccccccc} (\mathcal{N}_t^k K_t\{0, t - y\})_u & \xrightarrow{f_t^k(y)_u} & N_t^k(t - y)_u & \xrightarrow{g_t^k(y, u)} & N_t^k(t - u)_y & \xleftarrow{f_t^k(u)_y} & (\mathcal{N}_t^k K_t\{0, t - u\})_y \\ (\mathcal{N}_t^k p)_u \downarrow & & N_t^k(y, y')_u \downarrow & & x^{y' - y} \downarrow & & x^{y' - y} \downarrow \\ (\mathcal{N}_t^k K_t\{0, t - y'\})_u & \xrightarrow{f_t^k(y')_u} & N_t^k(t - y')_u & \xrightarrow{g_t^k(y', u)} & N_t^k(t - u)_{y'} & \xleftarrow{f_t^k(u)_{y'}} & (\mathcal{N}_t^k K_t\{0, t - u\})_{y'} \end{array}$$

Corollary 6.6. *Let F denote the projective resolution $S(-1) \rightarrow S$ of $K = S/\mathfrak{m}$ over S . Given $t, k \in \mathbb{N}$ with $0 \leq k \leq t + 1$ and $u, y \in [0, t]$, there is a chain of quasi-isomorphisms of chain complexes of K -vector spaces between $(T\mathcal{N}_t^{k+1} K_t\{t - u, t - u\})_y$ and $(F \otimes_S \mathcal{N}_t^k K_t\{0, t - y\})_u$. For each fixed u these quasi-isomorphisms are natural in y .*

Proof. We split up into the cases $u = 0$ and $u > 0$. First the case $u = 0$. The computation in Example 2.12 gives a quasi-isomorphism

$$(T\mathcal{N}_t^{k+1} K_t\{t, t\})_y \simeq (\mathcal{N}_t^k K_t\{0, t\})_y.$$

Lemma 6.4 implies that the latter chain complex is quasi-isomorphic to

$$(\mathcal{N}_t^k K_t\{0, t - y\})_0 = (F \otimes_S \mathcal{N}_t^k K_t\{0, t - y\})_0.$$

Next we treat the case $u > 0$. By exactness of the Nakayama functor and the computation in Example 2.12 we obtain quasi-isomorphisms

$$\begin{aligned} (T\mathcal{N}_t^{k+1} K_t\{t - u, t - u\})_y &\simeq (T\mathcal{N}_t^{k+1}(F \otimes_S K_t\{t - u, t\}))_y \\ &\simeq (\mathcal{N}_t^k(K_t\{0, t - u + 1\} \xrightarrow{p} K_t\{0, t - u\}))_y. \end{aligned}$$

By exactness of the Nakayama functor again the latter of these chain complexes is isomorphic to the mapping cone of the chain complex

$$(\mathcal{N}_t^k K_t\{0, t - u + 1\})_y \xrightarrow{(\mathcal{N}_t^k(p))_y} (\mathcal{N}_t^k K_t\{0, t - u\})_y.$$

Starting from the left side of the diagram of Corollary 6.5 we see that this map (notation is a bit switched, so $y' = u$) is quasi-isomorphic to the mapping cone of the chain complex to the chain complex

$$(\mathcal{N}_t^k K_t\{0, t - y\})_{u-1} \xrightarrow{x} (\mathcal{N}_t^k K_t\{0, t - y\})_u = (F \otimes_S \mathcal{N}_t^k(K_t\{0, t - y\}))_u.$$

The naturality statement follows from the fact that the K -vector spaces $T\mathcal{N}_t^{k+1} K_t\{t - u, t - u\}_y$ have homology concentrated in the same homological degree for all y . \square

6.2. General intervals. We shall now transfer the results for one dimensional intervals to general intervals. This is very easy because if $0 \leq \mathbf{r} \leq \mathbf{t}$ in \mathbb{N}^n then there is an isomorphism

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}} K_{\mathbf{t}}\{0, \mathbf{r}\} \cong \bigotimes_{j=1}^n \mathcal{N}_{t_j}^{k_j} K_{t_j}\{0, r_j\}$$

where the outer tensor product is over K , making it a module over $S = \bigotimes_{j=1}^n K[x_j]$.

With the notation $N_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r}) = \bigotimes_{j=1}^n N_{t_j}^{k_j}(r_j)$ there are obvious \mathbb{Z}^n -graded versions of all the results in Section 6.1. In particular Proposition 4.8 is the \mathbb{Z}^n -graded version of Corollary 6.3.

Proof of Proposition 2.16. First assume C is a module M . Choose a resolution $M \rightarrow E$ of M where each E_i is a direct sum of \mathbb{Z}^n -graded S -modules of the form $K_t\{0, \mathbf{r}\}$. Such a resolution clearly exists, it is an injective resolution of M ; see e.g. [8] for further details. By the \mathbb{Z}^n -graded version of Lemma 6.2, for each i , we have a quasi-isomorphism $\mathcal{N}_{\mathbf{t}}^{(t_j+2)\varepsilon_j} E_i \xrightarrow{\sim} T^{-2} E_i$, and therefore by naturality we obtain a quasi-isomorphism $\mathcal{N}_{\mathbf{t}}^{(t_j+2)\varepsilon_j} E \xrightarrow{\sim} T^{-2} E$. Since the Nakayama functor is exact, we are done.

Now assume C is any chain complex. It is then a colimit of bounded below chain complexes. Since the Nakayama functor commutes with colimits, is exact, and colimits is exact on chain complexes, we reduce to the case when C is bounded below. A bounded below complex is a colimit of truncations $\tau_{\leq p} C$ where

$$(\tau_{\leq p} C)_i = \begin{cases} 0 & i > p \\ \ker(C_p \rightarrow C_{p-1}) & i = p \\ C_i & i < p \end{cases}$$

Hence we may reduce to the case when C is bounded, and again by exactness of the Nakayama functor, we reduce to the case when C is a module M . \square

6.3. Connecting homomorphisms. Let $C' \xrightarrow{g} C$ be a morphism of chain complexes of S -modules. The *mapping cone* of g is the chain complex $C(g)$ with $C(g)_i = C'_{i-1} \oplus C_i$ and with differential sending (x, y) to $(-dx, gx + dy)$. If ι denotes the inclusion $\iota: C \rightarrow C(g)$ with $\iota(y) = (0, y)$, there is a short exact sequence $C \xrightarrow{\iota} C(g) \xrightarrow{\delta} TC'$ where $\delta(x, y) = x$. We call δ the *connecting homomorphism* associated to g .

Let M and N be S -modules. Considering M and N as chain complexes concentrated in degree zero there is for every pair of integers γ and ρ an isomorphism

$$\begin{aligned} \varphi: (T^\gamma M) \otimes_S C(g) \otimes_S (T^\rho N) &\rightarrow C((T^\gamma M) \otimes g \otimes (T^\rho N)) \\ m \otimes (x, y) \otimes n &\mapsto ((-1)^\gamma m \otimes x \otimes n, m \otimes y \otimes n) \end{aligned}$$

of chain complexes. Note that according to the Koszul Sign Convention or the conventions of [13], p.8 and 9, the differentials of the left and right complexes are given respectively by :

$$\begin{aligned} m \otimes (x, y) \otimes n &\mapsto (-1)^\gamma m \otimes (-dx, gx + dy) \otimes n \\ (m \otimes x \otimes n, m \otimes y \otimes n) &\mapsto ((-1)^\gamma m \otimes dx \otimes n, m \otimes gx \otimes n + (-1)^\gamma m \otimes dy \otimes n) \end{aligned}$$

We then have a commutative diagram :

$$\begin{array}{ccc}
 (T^\gamma M) \otimes_S C(g) \otimes_S (T^\rho N) & \xrightarrow{\varphi} & C((T^\gamma M) \otimes g \otimes (T^\rho N)) \\
 (T^\gamma M) \otimes \delta \otimes (T^\rho N) \downarrow & & \delta \downarrow \\
 (T^\gamma M) \otimes_S TC' \otimes_S (T^\rho N) & \xrightarrow{(-1)^\gamma} & T((T^\gamma M) \otimes_S C' \otimes_S (T^\rho N)).
 \end{array}$$

Now consider the inclusion

$$K_t\{k, t\} \xrightarrow{i} K_t\{0, t\}.$$

Note that the domain of i is $N_t^k(t - k)$ and that $N_t^k(t - k + 1)$ is the mapping cone of this morphism. Furthermore the map $N_t^k(k - 1, k)$ is the connecting map δ corresponding to the morphism i .

For arbitrary n , the map $N_t^k(\mathbf{y} - \epsilon_i, \mathbf{y})$ where $y_j = k_j$ may be identified as

$$\left(\bigotimes_{i < j} N_{t_i}^{k_i}(t_i - y_i) \right) \otimes \delta \otimes \left(\bigotimes_{i > j} N_{t_i}^{k_i}(t_i - y_i) \right)$$

which is then $(-1)^{\sum_{i=1}^{j-1} \gamma_i}$ times the connecting map of the injection

$$\begin{aligned}
 & \left(\bigotimes_{i < j} N_{t_i}^{k_i}(t_i - y_i) \right) \otimes K_t\{k_j, t_j\} \otimes \left(\bigotimes_{i > j} N_{t_i}^{k_i}(t_i - y_i) \right) \\
 \rightarrow & \left(\bigotimes_{i < j} N_{t_i}^{k_i}(t_i - y_i) \right) \otimes K_t\{0, t_j\} \otimes \left(\bigotimes_{i > j} N_{t_i}^{k_i}(t_i - y_i) \right).
 \end{aligned}$$

7. HOMOLOGICAL ALGEBRA OVER POSETS

In this section we give some relations between reduced cohomology groups of order complexes of various subposets of a poset P and Ext-groups of KP -modules.

7.1. Modules over posets. Let P be a partially ordered set. A KP -module is a functor from P , considered as a category with a morphism $p \rightarrow p'$ if and only if $p \leq p'$, to the category of K -vector spaces. Given two KP -modules M and N we denote the K -vector space of K -linear natural transformations from M to N by $\text{Hom}_{KP}(M, N)$. If A is a (commutative) K -algebra we may also consider AP -modules, that is, functors from P to the category of A -modules. If M is a KP -modules, and N an AP -module, then $\text{Hom}_{KP}(M, N)$ is naturally an A -module.

On occasion we allow M and N to be chain complexes of K -vector spaces. Then $\text{Hom}_{KP}(M, N)$ also becomes a chain complex, by taking the total complex of a double complex.

Definition 7.1. Let X be a convex subset of a poset P i.e. with the property that if $x, y \in X$ then all elements z with $x \leq z \leq y$ are in X . We define the KP -module $K\{X\}$ by letting $K\{X\}(p)$ be K if $p \in X$ and letting it be 0 otherwise. If $p \leq p'$ are both in X , then the structure homomorphism $K\{X\}(p) \rightarrow K\{X\}(p')$ is the identity on K . Otherwise this structure homomorphism is zero.

Remark 7.2. When P is finite the category of KP -modules is isomorphic to the category of (left) modules over the incidence algebra $I(P)$ of P over K . Given an order preserving map, that is, functor $f: P \rightarrow Q$ of partially ordered sets there is

an induced functor f^* from the category of KQ -modules to the category of KP -modules given by precomposition with f . However f does not in general induce a ring homomorphism from $I(P)$ to $I(Q)$. (Not even when $P = \{0 < 1\}$ and $Q = \{0\}$.) This is the main reason why we prefer to work with KP -modules instead of $I(P)$ -modules.

A subset J of P is an *order ideal* if $x \leq y$ in P and $y \in J$ implies $x \in J$. Dually an *order filter* in P is a subset F such that $y \leq x$ in P and $y \in F$ implies $x \in F$.

Lemma 7.3. *If an order ideal J in a poset P has a unique maximal element x , then $K\{J\}$ is an injective KP -module.*

Proof. This is immediate from the isomorphism

$$\mathrm{Hom}_{KP}(N, K\{J\}) \cong \mathrm{Hom}_K(N(x), K), \quad f \mapsto f(x).$$

□

Recall that the *order complex* of a partially ordered set P is the simplicial complex $\Delta(P)$ with the underlying set of P as vertex set, and a subset F of P is in $\Delta(P)$ if and only if it is a chain, that is, a totally ordered subset of P .

Proposition 7.4. *Let P be a poset. If E is a projective resolution of the KP -module $K\{P\}$ then there is an isomorphism*

$$H^* \mathrm{Hom}_{KP}(E, K\{P\}) \cong H^*(\Delta(P)).$$

Proof. Note first of all that the cohomology of $\mathrm{Hom}_{KP}(E, K\{P\})$ is independent of the choice of E . Also for p in P , the projective cover of $K\{p\}$ is the module E_p with $E_p(q) = K$ for $q \geq p$ and zero otherwise, and for $p \leq q \leq q'$ the morphism $E_p(q) \rightarrow E_p(q')$ is the identity.

For each k -chain in the poset P

$$c : p_0 < p_1 < \cdots < p_k$$

we get a projective $E_c = E_{p_k}$. Let E_k be the projective which is the direct sum of the E_c where c ranges over the k -chains. Then $E_k(q)$ has a basis consisting of the all $(k+1)$ -chains $p_0 < \cdots < p_k \leq q$. The E_k give a projective resolution of $K\{P\}$ with differential

$$(p_0 < p_1 < \cdots < p_k \leq q) \mapsto \sum_{i=0}^k (-1)^i (p_0 < \cdots < \hat{p}_i < \cdots < p_k \leq q).$$

This complex augmented with $K\{P\}$ is exact as K -vector spaces, since the map sending a $(k+1)$ -chain c to the $(k+2)$ -chain with q repeated if $p_k < q$ and to zero if $p_k = q$, gives a homotopy of the augmentation. Since $\mathrm{Hom}_{KP}(E_c, K\{P\})$ may be identified with the one-dimensional vector space with basis the dual of the chain c , we see that $\mathrm{Hom}_{KP}(E_k, K\{P\})$ is the vector space with basis the dual of all $(k+1)$ -chains, and so $\mathrm{Hom}_{KP}(E, K\{P\})$ is the cochain complex of the order complex $\Delta(P)$.

□

Lemma 7.5. *Let J be an order ideal in \mathbb{N}^n and let R denote the functor from J^{op} to \mathbb{Z}^n -graded S -modules with*

$$R(\mathbf{u}) = K\{0, \mathbf{u}\} := S/\mathfrak{m}^{\mathbf{u}+1}.$$

For every projective resolution $E \rightarrow K\{J^{\text{op}}\}$ of the J^{op} -module $K\{J^{\text{op}}\}$ the homomorphism $\text{Hom}_{KJ^{\text{op}}}(K\{J^{\text{op}}\}, R) \rightarrow \text{Hom}_{KJ^{\text{op}}}(E, R)$ is a quasi-isomorphism of chain complexes of \mathbb{Z}^n -graded S -modules.

Proof. It suffices to show that the above homomorphism is a quasi-isomorphism in every degree \mathbf{r} . Note that the KJ^{op} -module $\mathbf{u} \mapsto R(\mathbf{u})_{\mathbf{r}}$ is equal to the KJ^{op} -module $K\{((\mathbf{r} + \mathbb{N}^n) \cap J)^{\text{op}}\}$. Thus we have to show that the homomorphism

$$\text{Hom}_{KJ^{\text{op}}}(K\{J^{\text{op}}\}, K\{((\mathbf{r} + \mathbb{N}^n) \cap J)^{\text{op}}\}) \rightarrow \text{Hom}_{KJ^{\text{op}}}(E, K\{((\mathbf{r} + \mathbb{N}^n) \cap J)^{\text{op}}\})$$

is a quasi-isomorphism of chain complexes of K -vector spaces. This is a consequence of Lemma 7.3. \square

Remark 7.6. If F is an order filter in a poset P , then $\text{Hom}_{KP}(K\{F\}, N)$ is the (inverse) limit $\varprojlim_{p \in F} N(p)$. The order ideal J above corresponds to the monomial ideal I containing the monomials $x^{\mathbf{a}}$ such that \mathbf{a} is not in J . Then $\text{Hom}_{KJ^{\text{op}}}(K\{J^{\text{op}}\}, R)$ is simply equal to S/I .

7.2. Posets and cohomology of simplicial complexes. We shall need the following standard fact relating partially ordered sets and the geometric realization of their order complexes.

Proposition 7.7. *If P and Q are partially ordered sets and $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ are order preserving maps satisfying $\psi(\varphi(x)) \leq x$ and $\varphi(\psi(y)) \leq y$, then the induced maps $|\Delta(\varphi)|$ and $|\Delta(\psi)|$ of geometric realizations of order complexes are inverse homotopy equivalences.*

Proof. Consider the order preserving map $h: \{0 < 1\} \times P \rightarrow P$ with $h(0, x) = \psi(\varphi(x))$ and $h(1, x) = x$ and use the homeomorphism

$$|\Delta(\{0 < 1\} \times P)| \cong |\Delta(\{0 < 1\})| \times |\Delta(P)|.$$

\square

Given an order ideal J in \mathbb{N}^n and elements \mathbf{a} and \mathbf{b} in \mathbb{N}^n . Define the simplicial complex $\Delta_{\mathbf{a}}^{\mathbf{b}}(J)$ to consist of the subsets F of $\{1, 2, \dots, n\}$ such that the degree

$$\sum_{i \notin F} a_i \varepsilon_i + \sum_{i \in F} (b_i + 1) \varepsilon_i$$

is in J .

Lemma 7.8. *Given $\mathbf{a} \leq \mathbf{b}$ in \mathbb{N}^n , the geometric realization of the simplicial complex $\Delta_{\mathbf{a}}^{\mathbf{b}}(J)$ is homotopy equivalent to the geometric realization of the order complex of the poset $J \cap (\mathbf{a} + (\mathbb{N}^n \setminus [0, \mathbf{b} - \mathbf{a}]))$.*

Proof. Firstly the order poset of $\Delta_{\mathbf{a}}^{\mathbf{b}}(J)$ is isomorphic to the source of the inclusion

$$\varphi: J \cap (\mathbf{a} + \prod_{i=1}^n \{0, b_i - a_i + 1\}) \setminus \{\mathbf{a}\} \xrightarrow{\subseteq} J \cap (\mathbf{a} + (\mathbb{N}^n \setminus [0, \mathbf{b} - \mathbf{a}])).$$

It suffices to show that there is an order preserving map ψ in the opposite direction such that $\psi(\varphi(\mathbf{x})) = \mathbf{x}$ and $\varphi(\psi(\mathbf{y})) \leq \mathbf{y}$. In order to see this we note that the order preserving map

$$\psi: \mathbf{a} + (\mathbb{N}^n \setminus [0, \mathbf{b} - \mathbf{a}]) \rightarrow (\mathbf{a} + \prod_{i=1}^n \{0, b_i - a_i + 1\}) \setminus \{\mathbf{a}\}$$

with

$$\psi(\mathbf{y})_i = \begin{cases} a_i & \text{if } a_i \leq y_i \leq b_i \\ b_i + 1 & \text{if } b_i + 1 \leq y_i \end{cases}$$

is such a map. \square

Proposition 7.9. *Given an order ideal J in \mathbb{N}^n and $\mathbf{a} \leq \mathbf{b}$ in \mathbb{N}^n , and let $J_{\mathbf{a}}^{\mathbf{b}} = (J \cap [\mathbf{a}, \mathbf{b}])^{\text{op}}$.*

a. For every projective resolution E of the KJ^{op} -module $K\{J^{\text{op}}\}$ there is an isomorphism of the form

$$H^{i+1} \text{Hom}_{KJ^{\text{op}}}(E, K\{J_{\mathbf{a}}^{\mathbf{b}}\}) \cong \tilde{H}^i(\Delta_{\mathbf{a}}^{\mathbf{b}}(J)).$$

b. If $\mathbf{a}' \leq \mathbf{b}'$ in \mathbb{N}^n satisfy $\mathbf{a} \leq \mathbf{a}'$ and $\mathbf{b} \leq \mathbf{b}'$, then $\Delta_{\mathbf{a}}^{\mathbf{b}'}(J)$ and $\Delta_{\mathbf{a}'}^{\mathbf{b}}(J)$ are subcomplexes of $\Delta_{\mathbf{a}}^{\mathbf{b}}(J)$, there is an injection $K\{J_{\mathbf{a}}^{\mathbf{b}}\} \rightarrow K\{J_{\mathbf{a}'}^{\mathbf{b}'}\}$ and a projection $K\{J_{\mathbf{a}}^{\mathbf{b}}\} \rightarrow K\{J_{\mathbf{a}'}^{\mathbf{b}}\}$ of KJ^{op} -modules, and the diagrams

$$\begin{array}{ccc} H^{i+1} \text{Hom}_{KJ^{\text{op}}}(E, K\{J_{\mathbf{a}}^{\mathbf{b}}\}) & \xrightarrow{\cong} & \tilde{H}^i(\Delta_{\mathbf{a}}^{\mathbf{b}}(J)) \\ \downarrow & & \downarrow \\ H^{i+1} \text{Hom}_{KJ^{\text{op}}}(E, K\{J_{\mathbf{a}'}^{\mathbf{b}'}\}) & \xrightarrow{\cong} & \tilde{H}^i(\Delta_{\mathbf{a}'}^{\mathbf{b}'}(J)) \end{array}$$

and

$$\begin{array}{ccc} H^{i+1} \text{Hom}_{KJ^{\text{op}}}(E, K\{J_{\mathbf{a}}^{\mathbf{b}}\}) & \xrightarrow{\cong} & \tilde{H}^i(\Delta_{\mathbf{a}}^{\mathbf{b}}(J)) \\ \downarrow & & \downarrow \\ H^{i+1} \text{Hom}_{KJ^{\text{op}}}(E, K\{J_{\mathbf{a}'}^{\mathbf{b}}\}) & \xrightarrow{\cong} & \tilde{H}^i(\Delta_{\mathbf{a}'}^{\mathbf{b}}(J)) \end{array}$$

commute.

Proof. a. If $\mathbf{a} \notin J$, then $\Delta_{\mathbf{a}}^{\mathbf{b}}(J)$ is the empty simplicial complex \emptyset and both sides are zero. Otherwise we note that the partially ordered set $P = (J \cap (\mathbf{a} + \mathbb{N}^n))^{\text{op}}$ is the disjoint union of the order filter $F = (J \cap ([\mathbf{a}, \mathbf{b}]))^{\text{op}}$ and the order ideal $A = (J \cap (\mathbf{a} + (\mathbb{N}^n \setminus [0, \mathbf{b} - \mathbf{a}]))^{\text{op}}$. From the short exact sequence $K\{F\} \rightarrow K\{P\} \rightarrow K\{A\}$ we deduce that there is a short exact sequence

$$\text{Hom}_{KP}(E|_P, K\{F\}) \rightarrow \text{Hom}_{KP}(E|_P, K\{P\}) \rightarrow \text{Hom}_{KP}(E|_P, K\{A\}).$$

Since P has a unique maximal element we can use Lemma 7.3 to see that the cohomology of the middle term is a copy of K concentrated in cohomological degree zero. On the other hand, applying Proposition 7.4 we obtain isomorphisms

$$H^i \text{Hom}_{KP}(E|_P, K\{A\}) \cong H^i \text{Hom}_{KA}(E|_A, K\{A\}) \cong H^i(\Delta(A)).$$

Thus there is an isomorphism

$$\tilde{H}^i(\Delta(A)) \xrightarrow{\cong} H^{i+1} \text{Hom}_{KP}(E|_P, K\{F\})$$

for all i . Since P is an order ideal in J^{op} there is an isomorphism

$$H^{i+1} \text{Hom}_{KP}(E|_P, K\{F\}) \cong H^{i+1} \text{Hom}_{KJ^{\text{op}}}(E, K\{F\}).$$

Moreover there are isomorphisms

$$\tilde{H}^i(\Delta(A)) \cong \tilde{H}^i(\Delta(A^{\text{op}})) \cong \tilde{H}^i(\Delta(J \cap (\mathbf{a} + (\mathbb{N}^n \setminus [0, \mathbf{b} - \mathbf{a}]))).$$

The result now follows by applying Lemma 7.8. We leave the proof of the naturality statements in b. for the reader. \square

8. INDEPENDENCE OF \mathbf{t}

For a monomial ideal I there will be many \mathbf{t} such that I is positively \mathbf{t} -determined. This raises the question as to how our calculations depend on \mathbf{t} . In particular, what structural properties of the complexes $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)$ do not depend on \mathbf{t} and hence give intrinsic properties of the monomial ideal? In this section we consider such properties as vanishing of cohomology modules of these complexes and also linearity conditions on these complexes.

8.1. How complexes vary when \mathbf{t} vary. Since $\mathcal{N}_{\mathbf{t}}^1(M)$ essentially computes the local cohomology modules of M , it will have one non-vanishing cohomology module if and only if M is a Cohen-Macaulay module over S . Hence this is a property of the complex which is independent of which \mathbf{t} we chose.

We now present a different stability result for the Nakayama functor. Given k with $0 \leq k \leq t$ and $r \geq 0$ we let q_k^r denote the order preserving endomorphism of \mathbb{N} defined by the formula

$$q_k^r(i) = \begin{cases} i & \text{if } 0 \leq i \leq k \\ k & \text{if } k \leq i \leq k+r \\ i-r & \text{if } k+r \leq i. \end{cases}$$

With this notation it is readily seen that for $0 \leq k \leq t$ and $0 \leq y \leq t$ that

$$N_{t+r}^{k+1+r}(t-y) = (q_k^r)^* N_t^{k+1}(t-y).$$

By Lemma 6.2 we can conclude that if $n = 1$ and I is a t -determined ideal in $S = K[x]$, then the chain complexes $\mathcal{N}_{t+r}^{k+1+r}(S/I)$ and $(q_k^r)^* \mathcal{N}_t^{k+1}(S/I)$ are quasi-isomorphic.

Let us pass to the situation where $n > 1$. For $\mathbf{k}, \mathbf{r} \in \mathbb{N}^n$ with $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t}$ we define $q_{\mathbf{k}}^{\mathbf{r}}: \mathbb{N}^n \rightarrow \mathbb{N}^n$ by letting $q_{\mathbf{k}}^{\mathbf{r}}(\mathbf{x}) = (q_{k_1}^{r_1}(x_1), \dots, q_{k_n}^{r_n}(x_n))$.

Proposition 8.1. *Let M be a positively \mathbf{t} -determined S -module and let $\mathbf{k}, \mathbf{r} \in \mathbb{N}^n$ with $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t}$. There is a chain of quasi-isomorphisms between the chain complexes $\mathcal{N}_{\mathbf{t}+\mathbf{r}}^{\mathbf{k}+1+\mathbf{r}}(M)$ and $(q_{\mathbf{k}}^{\mathbf{r}})^* \mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}(M)$.*

Proof. First let us consider the case where $M = S/I$ for a positively \mathbf{t} -determined ideal I in S . Let J denote the order ideal in \mathbb{N}^n consisting of those $\mathbf{a} \in [0, \mathbf{t}]$ with $(S/I)_{\mathbf{a}} = K$. Let R denote the functor from J^{op} to the category of positively \mathbf{t} -determined S -modules with $R(\mathbf{w}) = K_{\mathbf{t}}\{0, \mathbf{w}\}$. Clearly, as an S -module

$$S/I \cong \lim_{\mathbf{w} \in J^{\text{op}}} R(\mathbf{w}) = \text{Hom}_{KJ^{\text{op}}}(K\{J\}, R).$$

We have seen in Lemma 7.5 that for $E \rightarrow K\{J\}$ a projective resolution of the KJ^{op} -module $K\{J\}$ there is a quasi-isomorphism

$$S/I \cong \text{Hom}_{KJ^{\text{op}}}(K\{J\}, R) \xrightarrow{\sim} \text{Hom}_{KJ^{\text{op}}}(E, R).$$

Since the Nakayama functor is exact it commutes with $\text{Hom}_{KJ^{\text{op}}}(E, -)$ in the sense that

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}(\text{Hom}_{KJ^{\text{op}}}(E, R)) \cong \text{Hom}_{KJ^{\text{op}}}(E, \mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1} \circ R).$$

One way to see this is by noting that $\text{Hom}_{KJ^{\text{op}}}(E, R)$ is the kernel of the homomorphism

$$\prod_{\mathbf{x} \in J} \text{Hom}_K(E(\mathbf{x}), R(\mathbf{x})) \rightarrow \prod_{(\mathbf{x} \geq \mathbf{y})} \text{Hom}_K(E(\mathbf{x}), R(\mathbf{y})),$$

taking the collection of maps $(f(\mathbf{x}))_{\mathbf{x} \in J}$ to the collection of maps $(g(\mathbf{x} \geq \mathbf{y}))_{(\mathbf{x} \geq \mathbf{y})}$, where $g(\mathbf{x} \geq \mathbf{y}): E(\mathbf{x}) \rightarrow R(\mathbf{y})$ is the map

$$g(\mathbf{x} \geq \mathbf{y}) = R(\mathbf{x} \geq \mathbf{y}) \circ f(\mathbf{x}) - f(\mathbf{y}) \circ E(\mathbf{x} \geq \mathbf{y}).$$

Since the Nakayama functor commutes with arbitrary direct products, this shows our claim. By Lemma 6.2 there is a quasi-isomorphism

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1} R(\mathbf{w}) = \mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1} K\{0, \mathbf{w}\} \xrightarrow{\sim} N_{\mathbf{t}}^{\mathbf{k}+1}(\mathbf{w}).$$

Thus we have provided a quasi-isomorphism

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}(S/I) \xrightarrow{\sim} \text{Hom}_{K^{J^{\text{op}}}}(E, N_{\mathbf{t}}^{\mathbf{k}+1}).$$

(Note that $N_{\mathbf{t}}^{\mathbf{k}+1}$ is a functor from J^{op} to the category of chain complexes of S -modules.) Similarly let J' be the $\mathbf{a} \in [0, \mathbf{t} + \mathbf{r}]$ such that $(S/I)_{\mathbf{a}} = K$. With E' the analog of E , there is a quasi-isomorphism

$$\mathcal{N}_{\mathbf{t}+\mathbf{r}}^{\mathbf{k}+\mathbf{r}+1}(S/I) \xrightarrow{\sim} \text{Hom}_{K^{J'^{\text{op}}}}(E', N_{\mathbf{t}+\mathbf{r}+1}^{\mathbf{k}+\mathbf{r}+1}).$$

The result now follows from the fact that $N_{\mathbf{t}+\mathbf{r}}^{\mathbf{k}+1+\mathbf{r}} = (q_{\mathbf{k}}^{\mathbf{r}})^* N_{\mathbf{t}}^{\mathbf{k}+1}$ for $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t}$, and that the structure maps in the above limits correspond to each other under this identification.

In the category of positively \mathbf{t} -determined S -modules, the indecomposable injectives are the $K_{\mathbf{t}}\{0, \mathbf{a}\}$ where $\mathbf{a} \in [0, \mathbf{t}]$. Note that these are of the form S/I . Any module in this category has a finite injective resolution. Since the Nakayama functor is exact and by Remark 2.15 it commutes with colimits and hence with direct sums, we get our result for any M . □

The above proposition may be stated in a different way, perhaps making the independence of \mathbf{t} more transparent. Recall from Proposition 2.16 that $\mathcal{N}_{\mathbf{t}}^{\mathbf{t}+2}$ is naturally quasi-isomorphic to the translation T^{-2n} . Since the Nakayama functors give auto-equivalences on derived categories, the following is natural.

Definition 8.2. For a positively \mathbf{t} -determined S -module M and $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t} + 2$ let

$$\mathcal{N}_{\mathbf{t}}^{-\mathbf{k}}(M) = T^{2n} \mathcal{N}_{\mathbf{t}}^{\mathbf{t}+2-\mathbf{k}}(M).$$

The following is then a reformulation of the previous proposition.

Proposition 8.3. *Let M be a finitely generated positively \mathbf{t} -determined S -module and let $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t} \leq \mathbf{t}'$ be multidegrees in \mathbb{N}^n . There is a chain of quasi-isomorphisms between the chain complexes $\mathcal{N}_{\mathbf{t}'}^{-\mathbf{k}-1}(M)$ and $(q_{\mathbf{t}-\mathbf{k}}^{\mathbf{t}'-\mathbf{t}})^* \mathcal{N}_{\mathbf{t}}^{-\mathbf{k}-1}(M)$.*

In particular for a monomial ideal I , the vanishing of a cohomology group of $\mathcal{N}_{\mathbf{t}}^{-\mathbf{k}-1}(S/I)$ is independent of which \mathbf{t} is chosen such that I is positively \mathbf{t} -determined, and is thus an intrinsic property of the monomial ideal.

Remark 8.4. In particular we would like to draw attention to the case when $\mathbf{k} = \mathbf{1}$. Then the vanishing of a cohomology group of $\mathcal{N}_{\mathbf{t}}^{-2}(S/I)$ is independent of \mathbf{t} for any $\mathbf{t} \geq \mathbf{1}$ such that S/I is positively \mathbf{t} -determined. In the case when S/I is square free, i.e. $\mathbf{1}$ -determined (it is then a Stanley-Reisner ring), the condition that $\mathcal{N}_{\mathbf{t}}^{-2}(S/I)$ has only one non-vanishing cohomology group is equivalent to S/I being Cohen-Macaulay. Thus this condition may provide a good(?) generalization to positively \mathbf{t} -determined monomial quotient rings, of the concept of a Stanley-Reisner ring being Cohen-Macaulay.

8.2. Linearity. The celebrated result of Eagon and Reiner stating that a simplicial complex is Cohen-Macaulay if and only if the square free monomial ideal of its Alexander dual complex has linear resolution, was generalized by Miller [8, Thm.4.20]. Let us recall his result. For a degree \mathbf{a} in \mathbb{N}^n let the support of \mathbf{a} , $\text{supp}(\mathbf{a})$, be the subset of $\{1, \dots, n\}$ consisting of those i for which $a_i > 0$. A \mathbb{N}^n -graded module M is said to have *support-linear* resolution if there is an integer d such that degree of every minimal generator of M has support of cardinality d , and $d \geq |\text{supp}(\mathbf{b})| - i$ for every multidegree \mathbf{b} and homological index i for which the Betti space $B_{i,\mathbf{b}}$ is nonzero. Miller shows the following.

Theorem 8.5. *Let M be a positively \mathbf{t} -determined module. The Alexander dual $A_{\mathbf{t}}(M)$ has support-linear resolution if and only if M is Cohen-Macaulay, i.e. $\mathcal{N}_{\mathbf{t}}^1(M)$ has only one non-vanishing cohomology module.*

We do not know of any other relationship between the vanishing of cohomology of some $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(M)$, and linearity conditions of some other $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}'}(M)$ or $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}'}A_{\mathbf{t}}(M)$. But there are notions of linearity of complexes which in the situation of $\mathcal{N}_{\mathbf{t}}^{-\mathbf{k}-1}(M)$ are independent of \mathbf{t} .

Definition 8.6. Let C be a chain complex of \mathbb{Z}^n -graded S -module and let $\mathbf{c} \in \mathbb{N}^n$. We say that C is \mathbf{c} -linear if there exists a number p_0 such that $B_p(C)_{\mathbf{b}} \neq 0$ implies that the cardinality of $\{j \mid b_j \geq c_j\}$ is

$$\begin{cases} n & \text{if } p \geq p_0 + n \\ p - p_0 & \text{if } p_0 \leq p \leq p_0 + n \\ 0 & \text{if } p \leq p_0. \end{cases}$$

Note that Millers notion of support-linear resolution is not a special case of \mathbf{c} -linearity.

Proposition 8.7. *Let $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t}$. Suppose $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}(M)$ is \mathbf{c} -linear.*

- a. *If $\mathbf{c} \leq \mathbf{k}$, then $\mathcal{N}_{\mathbf{t}+\mathbf{r}}^{\mathbf{k}+1+\mathbf{r}}(M)$ is \mathbf{c} -linear for any \mathbf{r} in \mathbb{N}^n .*
- b. *If $\mathbf{c} \geq \mathbf{k}$, then $\mathcal{N}_{\mathbf{t}+\mathbf{r}}^{\mathbf{k}+1+\mathbf{r}}(M)$ is $\mathbf{c} + \mathbf{r}$ -linear for any \mathbf{r} in \mathbb{N}^n .*

Proof. By Proposition 8.1 the Betti spaces

$$B_p(\mathcal{N}_{\mathbf{t}+\mathbf{r}}^{\mathbf{k}+1+\mathbf{r}}(M))_{\mathbf{d}} = B_p(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}+1}(M))_{q_{\mathbf{k}}^{\mathbf{r}}(\mathbf{d})}.$$

Then a. follows because coordinate i in \mathbf{d} is $< c_i$ if and only if the same holds in $q_{\mathbf{k}}^{\mathbf{r}}(\mathbf{d})$. Part b. follows because coordinate i in \mathbf{d} is $\geq c_i + r_i$ if and only if coordinate i in $q_{\mathbf{k}}^{\mathbf{r}}(\mathbf{d})$ is $\geq c_i$. \square

Alternatively this may be stated as.

Proposition 8.8. *Let $\mathbf{0} \leq \mathbf{k} \leq \mathbf{t} \leq \mathbf{t}'$. Suppose $\mathcal{N}_{\mathbf{t}}^{-\mathbf{k}-1}(M)$ is $\mathbf{t} - \mathbf{c}$ -linear.*

- a. *If $\mathbf{c} \geq \mathbf{k}$, then $\mathcal{N}_{\mathbf{t}'}^{-\mathbf{k}-1}(M)$ is $\mathbf{t} - \mathbf{c}$ -linear.*
- b. *If $\mathbf{c} \leq \mathbf{k}$, then $\mathcal{N}_{\mathbf{t}'}^{-\mathbf{k}-1}(M)$ is $\mathbf{t}' - \mathbf{c}$ -linear.*

In the pivotal case of Proposition 8.8, when $\mathbf{c} = \mathbf{k}$, and the polynomial ring S is in two variables, we have worked out explicitly what it means for $\mathcal{N}^{-\mathbf{k}-1}(S/I)$ to be $\mathbf{t} - \mathbf{k}$ -linear. The reader may wish to compare this with the results in Section 5 concerning the vanishing of cohomology modules of these complexes, when we have two variables. We state the following without proof.

Proposition 8.9. *Suppose S is the polynomial ring in two variables, and let $\mathbf{1} \leq \mathbf{k} \leq \mathbf{t}$. Then $\mathcal{N}_{\mathbf{t}}^{-\mathbf{k}-1}(S/I)$ is $\mathbf{t} - \mathbf{k}$ -linear if and only if either*

- a. I is \mathbf{k} -determined.*
- b. I is $(x^a y^b)$ where $(a, b) \geq \mathbf{k}$.*
- c. I is $(x^{a_0}, x^{a_1} y^{b_1}, \dots, x^{a_{p-1}} y^{b_{p-1}}, y^{b_p})$ where all $t_1 \geq a_i \geq k_1$ and $t_2 \geq b_i \geq k_2$.*

9. PROOFS OF THE MAIN THEOREMS

In this section we prove Theorem 4.4, Theorem 4.6 and Theorem 4.10. Originally our approach was to use a homotopy limit spectral sequence argument, see for instance [12]. However the spectral sequences degenerated so nicely that we are able to do without them. Let us fix $\mathbf{k}, \mathbf{t} \in \mathbb{N}^n$ with $\mathbf{k} \leq \mathbf{t} + \mathbf{1}$.

Proof of Theorem 4.4. Let J denote the order ideal in \mathbb{N}^n consisting of those $\mathbf{a} \in [0, \mathbf{t}]$ with $(S/I)_{\mathbf{a}} = K$. Let R denote the functor from J^{op} to the category of positively \mathbf{t} -determined S -modules with $R(\mathbf{w}) = K_{\mathbf{t}}\{0, \mathbf{w}\}$. Clearly, as an S -module

$$S/I \cong \lim_{\mathbf{w} \in J^{\text{op}}} R(\mathbf{w}) = \text{Hom}_{K J^{\text{op}}}(K\{J^{\text{op}}\}, R).$$

We have seen in Lemma 7.5 that for $E \rightarrow K\{J^{\text{op}}\}$ a projective resolution of the $K J^{\text{op}}$ -module $K\{J^{\text{op}}\}$ there is a quasi-isomorphism

$$S/I \cong \text{Hom}_{K J^{\text{op}}}(K\{J^{\text{op}}\}, R) \xrightarrow{\sim} \text{Hom}_{K J^{\text{op}}}(E, R).$$

Since the Nakayama functor is exact it commutes with $\text{Hom}_{K J^{\text{op}}}(E, -)$ in the sense that

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(\text{Hom}_{K J^{\text{op}}}(E, R)) \cong \text{Hom}_{K J^{\text{op}}}(E, \mathcal{N}_{\mathbf{t}}^{\mathbf{k}} \circ R),$$

as we explained in the first part of the proof of Proposition 8.1. For \mathbf{w} in J there is by Lemma 6.2 a quasi-isomorphism

$$(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}} R(\mathbf{w}))_{\mathbf{r}} = (\mathcal{N}_{\mathbf{t}}^{\mathbf{k}} K_{\mathbf{t}}\{0, \mathbf{w}\})_{\mathbf{r}} \xrightarrow{\sim} (N_{\mathbf{t}}^{\mathbf{k}}(\mathbf{w}))_{\mathbf{r}}.$$

Moreover by Lemma 6.4 there is a quasi-isomorphism

$$(N_{\mathbf{t}}^{\mathbf{k}}(\mathbf{w}))_{\mathbf{r}} \xrightarrow{\sim} (N_{\mathbf{t}}^{\mathbf{k}}(\mathbf{t} - \mathbf{r}))_{\mathbf{t} - \mathbf{w}},$$

and this quasi-isomorphism is natural in \mathbf{w} , by Corollary 6.5. Let $\gamma = \gamma_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$, let $\mathbf{u} = \mathbf{u}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$ and let $\mathbf{v} = \mathbf{v}_{\mathbf{t}}^{\mathbf{k}}(\mathbf{r})$. By Proposition 4.8 and Remark 4.5, there is a quasi-isomorphism

$$(N_{\mathbf{t}}^{\mathbf{k}}(\mathbf{t} - \mathbf{r}))_{\mathbf{t} - \mathbf{w}} \xrightarrow{\sim} T^{-\gamma} K_{\mathbf{t}}\{\mathbf{t} - \mathbf{v}, \mathbf{t} - \mathbf{u}\}_{\mathbf{t} - \mathbf{w}} = T^{-\gamma} K\{[\mathbf{u}, \mathbf{v}]^{\text{op}}\}(\mathbf{w}),$$

where $K\{[\mathbf{u}, \mathbf{v}]^{\text{op}}\}$ is a module as in Definition 7.1. Since $\mathbf{w} \in J^{\text{op}}$ the latter is $T^{-\gamma} K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\}(\mathbf{w})$. Summing up we have a quasi-isomorphism

$$(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}} R(\mathbf{w}))_{\mathbf{r}} \xrightarrow{\sim} T^{-\gamma} K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\}(\mathbf{w}).$$

which is natural in \mathbf{w} in the sense that if $\mathbf{w} \leq \mathbf{w}'$ then the corresponding diagram commutes. Thus we have a quasi-isomorphism

$$\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \xrightarrow{\sim} \text{Hom}_{K J^{\text{op}}}(E, T^{-\gamma} K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\}).$$

Finally applying Proposition 7.9 we obtain isomorphisms

$$H^i \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}(S/I)_{\mathbf{r}} \cong H^{i-\gamma}(\text{Hom}_{K J^{\text{op}}}(E, K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\})) \cong \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}(J)).$$

□

Proof of Theorem 4.6. The functor $F \otimes_S \mathcal{N}_t^k$ is exact, and therefore it commutes with $\text{Hom}_{K^{J^{\text{op}}}}(E, -)$. By Corollary 6.6 there is a quasi-isomorphism

$$(F \otimes_S \mathcal{N}_t^k R(\mathbf{w}))_{\mathbf{r}} = (F \otimes_S \mathcal{N}_t^k K_t\{0, \mathbf{w}\})_{\mathbf{r}} \simeq T^n \mathcal{N}_t^{k+1}(K_t\{\mathbf{t} - \mathbf{r}, \mathbf{t} - \mathbf{r}\})_{\mathbf{t}-\mathbf{w}},$$

and this quasi-isomorphism is natural in \mathbf{w} . Let γ , \mathbf{a} and \mathbf{b} be such that there is a quasi-isomorphism

$$\mathcal{N}_t^{k+1}(K_t\{\mathbf{t} - \mathbf{r}, \mathbf{t} - \mathbf{r}\}) \simeq T^{-\gamma-n} K_t\{\mathbf{t} - \mathbf{b}, \mathbf{t} - \mathbf{a}\}.$$

Note that since $\mathbf{w} \in J$ we have

$$K_t\{\mathbf{t} - \mathbf{b}, \mathbf{t} - \mathbf{a}\}_{\mathbf{t}-\mathbf{w}} = K\{(J \cap [\mathbf{a}, \mathbf{b}])^{\text{op}}\}(\mathbf{w}).$$

Summing up we have a quasi-isomorphism

$$(F \otimes_S \mathcal{N}_t^k R(\mathbf{w}))_{\mathbf{r}} \simeq T^{-\gamma} K\{(J \cap [\mathbf{a}, \mathbf{b}])^{\text{op}}\}(\mathbf{w}),$$

which is natural in \mathbf{w} in the sense that if $\mathbf{w} \leq \mathbf{w}'$ then the corresponding diagram commutes. Thus we have a quasi-isomorphism

$$F \otimes_S \mathcal{N}_t^k(S/I)_{\mathbf{r}} \xrightarrow{\simeq} \text{Hom}_{K^{J^{\text{op}}}}(E, T^{-\gamma} K\{(J \cap [\mathbf{a}, \mathbf{b}])^{\text{op}}\}).$$

Finally applying Proposition 7.9 we obtain isomorphisms

$$H^i(F \otimes_S \mathcal{N}_t^k(S/I)_{\mathbf{r}}) \cong H^{i-\gamma}(\text{Hom}_{K^{J^{\text{op}}}}(E, K\{(J \cap [\mathbf{a}, \mathbf{b}])^{\text{op}}\})) \cong \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{a}}^{\mathbf{b}}(J)).$$

□

Proof of Theorem 4.10. We shall describe the homomorphism

$$H^i(\mathcal{N}_t^k S/I)_{\mathbf{r}-\varepsilon_j} \rightarrow H^i(\mathcal{N}_t^k S/I)_{\mathbf{r}}$$

for every i and every \mathbf{r} with $\varepsilon_j \leq \mathbf{r} \leq \mathbf{t}$. In order to describe this homomorphism we need to take a closer look at the maps in the proof of Theorem 4.4. Let us continue with the notation introduced in that proof. We split up into the three cases where $k_j < r_j$, where $k_j = r_j$ and where $k_j > r_j$.

Firstly if $k_j < r_j$, the natural homomorphism $\mathcal{N}_t^k R(\mathbf{w})_{\mathbf{r}-\varepsilon_j} \rightarrow \mathcal{N}_t^k R(\mathbf{w})_{\mathbf{r}}$ of functors in $\mathbf{w} \in J^{\text{op}}$ corresponds to the homomorphism of $K^{J^{\text{op}}}$ -modules

$$T^{-\gamma} K\{(J \cap [\mathbf{u} - \varepsilon_j, \mathbf{v}])^{\text{op}}\} \rightarrow T^{-\gamma} K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\}.$$

Applying $T \text{Hom}_{K^{J^{\text{op}}}}(E, -)$ to this homomorphism we obtain the homomorphism

$$(\mathcal{N}_t^k(S/I))_{\mathbf{r}-\varepsilon_j} \rightarrow (\mathcal{N}_t^k(S/I))_{\mathbf{r}}.$$

The second naturality statement of Proposition 7.9 gives that the associated homomorphism of i 'th cohomology groups is isomorphic to the homomorphism of $i - \gamma - 1$ 'th reduced cohomology groups coming from the inclusion of spaces $\Delta_{\mathbf{u}}^{\mathbf{v}} \subseteq \Delta_{\mathbf{u}-\varepsilon_j}^{\mathbf{v}}$.

Next we consider the case $k_j = r_j$. Consider the map

$$(\mathcal{N}_t^k \circ R)(\mathbf{w})_{\mathbf{r}-\varepsilon_j} \rightarrow (\mathcal{N}_t^k \circ R)(\mathbf{w})_{\mathbf{r}}.$$

As explained in the proof of Theorem 4.4, it corresponds to the natural map

$$N_t^k(\mathbf{r} - \varepsilon_j, \mathbf{r})_{\mathbf{t}-\mathbf{w}} : N_t^k(\mathbf{t} - \mathbf{r} + \varepsilon_j)_{\mathbf{t}-\mathbf{w}} \rightarrow N_t^k(\mathbf{t} - \mathbf{r})_{\mathbf{t}-\mathbf{w}}.$$

The right side of the map above is $T^{-\gamma} K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\}(\mathbf{w})$, the identification being natural in \mathbf{w} . For slim notation let

$$\alpha = u_{t_j}^{k_j}(r_j - 1) - u_{t_j}^{k_j}(r_j) = t_j + 1 - k_j, \quad \beta = v_{t_j}^{k_j}(r_j - 1) - v_{t_j}^{k_j}(r_j) = k_j$$

By Subsection 6.3 the above map, considered as depending functorially on \mathbf{w} , is $(-1)^{\sum_{i=1}^{j-1} \gamma_{t_i}^{k_i}(r_i)}$ times the connecting homomorphism of the first map \hat{i} in the short exact sequence

$$\begin{aligned} 0 &\rightarrow T^{-\gamma-1}K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\} \xrightarrow{\hat{i}} T^{-\gamma-1}K\{(J \cap [\mathbf{u}, \mathbf{v} + \beta\varepsilon_j])^{\text{op}}\} \\ &\rightarrow T^{-\gamma-1}K\{(J \cap [\mathbf{u} + \alpha\varepsilon_j, \mathbf{v} + \beta\varepsilon_j])^{\text{op}}\} \rightarrow 0. \end{aligned}$$

We now apply $T \text{Hom}_{KJ^{\text{op}}}(E, -)$ to this short exact sequence, and obtain a new short exact sequence of complexes. Note the following three things.

1. The new short exact sequence is by Proposition 7.9 quasi-isomorphic to the (non-exact) sequence of homologically shifted reduced cochain complexes associated to the inclusions

$$\Delta_{\mathbf{u}+\alpha\varepsilon_j}^{\mathbf{v}+\beta\varepsilon_j} \subseteq \Delta_{\mathbf{u}}^{\mathbf{v}+\beta\varepsilon_j} \subseteq \Delta_{\mathbf{u}}^{\mathbf{v}}.$$

The simplicial complex $\Delta_{\mathbf{u}+\alpha\varepsilon_j}^{\mathbf{v}}$ is contractible because it is a cone on the vertex j , that is, it contains F if and only if it contains $F \cup \{j\}$. Since we are in the situation of Lemma 4.2 the above short exact sequence is quasi-isomorphic to the short exact sequence

$$\tilde{C}^*(\Delta_{\mathbf{u}}^{\mathbf{v}}) \rightarrow \tilde{C}^*(\Delta_{\mathbf{u}}^{\mathbf{v}+\beta\varepsilon_j}) \oplus \tilde{C}^*(\Delta_{\mathbf{u}+\alpha\varepsilon_j}^{\mathbf{v}}) \rightarrow \tilde{C}^*(\Delta_{\mathbf{u}+\alpha\varepsilon_j}^{\mathbf{v}+\beta\varepsilon_j})$$

of reduced cochain complexes underlying the Mayer–Vietoris cohomology exact sequence.

2. The connection homomorphism for this short exact sequence

$$\delta^{i-\gamma-2}: \tilde{H}^{i-\gamma-2}(\Delta_{\mathbf{u}+\alpha\varepsilon_j}^{\mathbf{v}+\beta\varepsilon_j}) \rightarrow \tilde{H}^{i-\gamma-1}(\Delta_{\mathbf{u}}^{\mathbf{v}}).$$

identifies as the morphism on homology we get by applying $T \text{Hom}_{KJ^{\text{op}}}(E, -)$ to the connecting homomorphism of \hat{i} .

3. If we apply $T \text{Hom}_{KJ^{\text{op}}}(E, -)$ to the morphism of KJ^{op} -modules

$$(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}} \circ R)(\cdot)_{\mathbf{r}-\varepsilon_j} \rightarrow (\mathcal{N}_{\mathbf{t}}^{\mathbf{k}} \circ R)(\cdot)_{\mathbf{r}}$$

and take homology this gives the map

$$H^i(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}S/I)_{\mathbf{r}-\varepsilon_j} \rightarrow H^i(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}S/I)_{\mathbf{r}}.$$

Hence this map is isomorphic to $(-1)^{\sum_{i=1}^{j-1} \gamma_{t_i}^{k_i}(r_i)}$ times the Mayer–Vietoris connecting homomorphism.

Finally we consider the case $k_j > r_j$. In this case the natural homomorphism $\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}R(\mathbf{w})_{\mathbf{r}-\varepsilon_j} \rightarrow \mathcal{N}_{\mathbf{t}}^{\mathbf{k}}R(\mathbf{w})_{\mathbf{r}}$ of functors in $\mathbf{w} \in J^{\text{op}}$ corresponds to the homomorphism

$$T^{-\gamma}K\{(J \cap [\mathbf{u}, \mathbf{v} - \varepsilon_j])^{\text{op}}\} \rightarrow T^{-\gamma}K\{(J \cap [\mathbf{u}, \mathbf{v}])^{\text{op}}\}.$$

Applying $T \text{Hom}_{KJ^{\text{op}}}(E, -)$ to this homomorphism we obtain the homomorphism

$$(\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}S/I)_{\mathbf{r}-\varepsilon_j} \rightarrow (\mathcal{N}_{\mathbf{t}}^{\mathbf{k}}S/I)_{\mathbf{r}}.$$

By the first naturality part of Proposition 7.9 this is quasi-isomorphic to the homomorphism of reduced cochain complexes induced by the inclusion $\Delta_{\mathbf{u}}^{\mathbf{v}} \subseteq \Delta_{\mathbf{u}}^{\mathbf{v}-\varepsilon_j}$. \square

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